DEVELOPMENT OF A QUADTREE BASED AGGLOMERATION METHOD FOR UNSTRUCTURED HYBRID GRIDS

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ABSTRACT

In this study, a new cell agglomeration technique is developed and applied to viscous multigrid flow solutions on unstructured/hybrid grids. The grid coarsening required by the multigrid scheme is achieved by agglomerating the unstructured/hybrid cells based on their distribution on a quadtree data structure. The paper presents the coarsening strategy and the multigrid adaptation on inviscid/viscous 2D solutions over a NACA 0012 airfoil section. It is shown that, the quadtree based agglomeration and grid coarsening provides well defined, nested, body fitted and optimum aspect ratio cells at all coarse grid levels. In the cases studies, it is observed that, the multigrid flow solutions obtained in this study provide convergence accelerations about 10 times for low speed inviscid flows and in a range of 5 to 30 fold for viscous flows.

INTRODUCTION

The multigrid (MG) technique is likely to be the most effective technique to achieve the reduction the CPU cost of a single computation for flow solvers [5][1]. The basic idea of multigrid strategy is to employ multilevel grids which are the coarsening subsets of the original grid. An explicit solver rapidly reduces the high frequency errors on the computational grids. In a MG solution high frequency errors at each grid level are therefore reduced rapidly. Since high frequency errors on coarse grids correspond to low frequency errors on fine grids, cycling through the coarse grid levels rapidly reduces all the errors ranging from high to low frequency on the original fine grid. 

In MG applications on structured grids, a coarse grid can easily be derived from a given fine grid by omitting every other point in each coordinate direction. Recursive application of this procedure results in a sequence of coarse grids. The main difficulty with unstructured multigrid methods is the construction of the coarse grid levels. A variety of techniques have been proposed for unstructured multigrid coarse grid constructions [9]. Agglomeration technique is a widely used method due to being fully nested, easily automated, no geometry loss and high solution accuracy [8]. In an agglomeration grid cells are fused together to form a smaller set of larger polygonal (or polyhedral in three dimensions) control volumes. The main difficulty in agglomeration is the selection of the cells to be agglomerated so that the new cells formed have acceptable aspect ratios. [9]

In this study, a new grid coarsening technique is developed. This technique relies on agglomerating unstructured/hybrid cells based on their distribution on a quadtree data structure [10]. It is shown that the new grid agglomeration method provides well defined, nested, body fitted and optimum aspect ratio cells at all coarse grid levels. The method is integrated into the flow solver SENSE2D and applied for the solution of 2D viscous flows over NACA0012 airfoil section using various multigrid cycling strategies such as classical V cycle and Full Multigrid. It is observed that, multigrid flow solutions provide convergence accelerations in a range 5 to 30 fold in the flow cases studied.

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METHOD

The basic idea of MG strategy is to employ multilevel grids which are the coarser subsets of the original grid. The main objective of this study is to create coarser subsets of the original grid by cell agglomeration. On structured grids, a coarser grid can easily be obtained from a given fine mesh by omitting every other point in each coordinate direction. However, on unstructured grids, the construction of coarser grid levels is not an easy task and a variety of techniques have been proposed for coarse grid constructions [5], [2], [4], [6], [9]. Cell agglomeration techniques are widely used, due to being fully nested, easily automated, without geometry loss and with high solution accuracy [8]. In this method, the methodology of selecting the cells to be agglomerated is critical in keeping the aspect ratio of the cells acceptable [9]. In this study, a new grid coarsening method which is based on representation of the grid cells in a quadtree hierarchical data structure.

In MG solution method, the flow solution is mostly based on the flux computation on the active edges and at each coarse level, active edges are already defined in the agglomeration process. Therefore, the baseline flow solver, SENSE2D, is modified to proceed the flux calculation along the edges and the data structure and solution algorithm are modified to accommodate the edge based solution method. The modifications are summarized at “Modification on SENSE2D Solver” section.

Finally, the MG routines are prepared and embedded to the baseline code. The Full Approximation Storage Scheme (FAS) is chosen to directly handle non-linear problems. In FAS, flow variables themselves and the solution residuals at each level are transferred to the coarser level, which is known as a restriction process. The corrections obtained at the coarser levels are then transferred back to the finer level grid solution which is known as a prolongation process. The multigrid strategies known as V-cycle and Full Multigrid cycle are implemented. The MG algorithm is summarized at the “Multigrid Adaptation” section.

Grid Coarsening Based on Quadtree Hierarchy

Quadtree or octree data structures are well known in storing data associated with location in 2 and 3-dimensional space, respectively [10]. The term quadtree is used to describe a hierarchical data structure which is based on the successive subdivision of the space into four equal-sized quadrants or childs. In this study, a quadtree data structure is used for first grouping a maximum of four cells together in a quadrant, and then merging the cells in a quadrant with the cells in the parent quadrant successively. A computational grid (Figure 1-a) is first represented by the nodes corresponding to the cell centers (Figure 1-b); the nodes are placed in the quadtree structure. The quadrants formed by a maximum of four nodes are shown in Figure 1-c, d. It should be noted that the coarse grid cells formed by agglomerating the neighboring cells in such a way provide an aspect ratio of about one. The coarser grids are obtained by agglomerating the cells in the quadrants into the parent quadrants successively (Figure 1-e). The coarsest grid is shown in Figure 1-f. In the grid coarsening process, the coarser grids are nested, that is, there is no a newly generated edge or nodes between the grid levels. Such a property also assures that the solid boundaries are not removed at the coarser grid levels.
Modification on Flow Solver

The viscous flow solver, SENSE2D, is taken as the baseline flow solver. It is an unstructured FVM solver, flow variables are stored at cell centers and second order Roe’s upwind flux computations are employed. The time dependent equations are solved explicitly using the third order Runge-Kutta method with variable time-stepping [9]. The solution algorithm proceeds along the cells and the flux computations are carried out along the edges of each triangular/quadrilateral cell. In an MG solution method, since the cells are no longer regular cells and may have many number of edges. The solution algorithm therefore proceeds along the active edges rather than the cells. The active edges in each grid level are already defined in the agglomeration process. In this study the data structure and solution algorithm are therefore modified to accommodate an edge based solution method rather than a cell based one. An edge based solution algorithm can easily accommodate varying cell structures in MG levels. The implementation of boundary conditions is kept the same, since the boundary edges are not removed in the agglomeration process.
Multigrid Adaptation

During MG adaptation, Full Approximation Storage Scheme (FAS) is chosen to directly handle non-linear problems [5], [8]. In FAS, flow variables themselves and residual at each level are transferred to the coarser levels, the process is known as restriction. The corrections obtained at the coarser levels are then transferred back to the finer level grid solution which is known as prolongation. As a final important point, V-cycles and Full Multigrid cycles are embedded to the code.

**Full Approximation Storage Scheme:** The discrete problem at the fine grid level is defined;

\[ L_h u_h = 0 \quad (1) \]

where, subscript represents the grid level, \( L \) is nonlinear differential space operator, \( u_h \) is the converged discrete solution. For an unconverged solution in the fine level, by using the current estimate of the solution by iterative technique, \( \bar{u}_h \), Equation (1) is not satisfied exactly, therefore;

\[ L_h \bar{u}_h = r_h \quad (2) \]

\[ L_h \bar{u}_h - L_h \bar{u}_h = -r_h \quad (3) \]

where \( r_h \) is the residual of fine grid level. At the coarser grid level, \( L_H \bar{u}_H = 0 \) similarly to Equation (1), for the unconverged solution at the coarser level, Equation (3) can be written as,

\[ L_H \bar{u}_H - L_H \bar{u}_H = -r_H \quad (4) \]

where \( r_H \) is the residual of coarse grid level. It can be seen that the exact solution can be transferred as,

\[ L_H \bar{u}_H = I_H \bar{u}_H \quad (5) \]

and both sides give zero. If we substitute fine level exact solution to coarse level equation by using this equality given in Equation (5), in the coarse level, it can be obtained as,

\[ L_H \bar{u}_H = L_H I_H \bar{u}_H - I_H r_h + r_H \quad (6) \]

Since the first two terms on right-hand side are constant and transferred from the fine level solution, they are sometimes called the forcing function for the coarse level solution.

Once the coarse grid equations are solved, the corrections on the flow variables are prolonged to the fine grid level:

\[ \bar{u}_h^{new} = \bar{u}_h^{old} + I_H (\bar{u}_H^{new} - I_H \bar{u}_H^{old}) \quad (7) \]

**Transfer Operators:** One of the key elements for the success of such a method is the development of efficient transfer mechanisms for flow variables and residuals between grids, which are known as restriction \( I_H \) and prolongation process \( I_H \). In the restriction of flow variables an area weighted averaging is used:

\[ I_H (u_h) = \frac{\sum A_{fine} u_h}{\sum A_{coarse}} \quad (8) \]

The residuals are restricted by simple summation of the fine level residuals:

\[ I_H (r_h) = \sum r_h \quad (9) \]

The corrections on the flow variables on the coarse level cells are prolonged to the fine grid level \( I_H \) by simply assigning the corrections to all the fine level cells forming the coarse cell.

**Cycling:** The V-cycles and Full MG cycle patterns are applied to the MG application. The V-cycle begins on the finest grid of the sequence, where one relaxation or time-step is performed. The solution and residuals are then interpolated to the next coarser grid, where another time-step is performed. This procedure is repeated on each coarser grid until the coarsest grid of the sequence is reached.
Then refinement phase starts and the coarse grid corrections are prolonged back to each successively finer grid. At the classical V-cycle strategy, single or multiple time-steps on each grid level is performed (Figure 2-a). This refinement procedure is repeated until the finest grid of the sequence is reached. The combination of grid sequencing with a multigrid method (where the solution on the current grid is initiated from a previously computed solution on a coarser grid) results in a strategy known as the full multigrid procedure. The basic cycling strategy is depicted in Figure 2-b. Beginning with an initial sequence of grids, the solution on the finest grid of the sequence is obtained by a multigrid procedure operating on this sequence. A new finer grid is then added to the sequence, the solution is interpolated onto this new grid, and MG cycling is resumed on this new larger sequence of grids. The procedure can be repeated, each time adding a new finer grid to the sequence, until the desired solution accuracy has been achieved, or the finest available grid has been reached.

RESULTS AND DISCUSSION

In this section we present results for a set of inviscid and viscous flows in two dimensional flows. The flows are computed over NACA0012 airfoil with \( M=0.01/\alpha=0^\circ \) and \( M=0.1/\alpha=3^\circ \). A sequence of four coarse hybrid grids is generated using the quadtree based agglomeration method. In the solution process, FMG and the V-cycle strategies are applied with 20 time steps at each grid level.

Grid Coarsening and Cell Agglomeration

An automated quadtree based data structure is used for coarsening the unstructured hybrid grid around NACA0012 airfoil. The fine grid level, shown in Figure 3, contains 42,227 nodes and 65,540 tetrahedrals / quadriterals. Coarser grids for MG levels are shown in Figure 4-a to Figure 4-d with their sizes. In this case, maximum of 60% coarsening ratio between the coarse grid levels is used. The connectivity information between grids is obtained from the data tree via this parent / child relationship. It should be noted that the quadtree based coarser grids have good quality cells with aspect ratios of about one and the coarsening gets more significant at the far field boundaries.
Figure 4 NACA 0012 grid levels (close up view)
Case 1

The first test case involving an inviscid flow over NACA0012 airfoil is at a low Mach number of 0.01, a Reynolds number of $10^5$ and at 0° angle of attack. The solution is obtained using the second order flux computations with 0.1 CFL number. The computed pressure distribution around the airfoil is presented in Figure 5. The same solution is then obtained by applying V-cycle MG and Full MG algorithms. The convergence histories of the solutions in terms of the variation of the residual and the aerodynamic loads are given in Figures 6 and 7. The residual is defined as the L2 norm of the conservation of mass equation. It is seen in Figure 6 the FMG and V-cycle MG solutions exhibit about the same converge rate as expected and they are approximately 10 times faster than the single grid solution without the MG. Such convergence acceleration is in agreement with the findings in literature [2].
A similar convergence rate is observed in the variation of drag, lift and pitching moment coefficients when the solutions with and without MG are compared as shown in Figure 7. It should be noted that, FMG and V-cycle solutions have a monotone convergence history.

![Figure 7 Variation of aerodynamic coefficients in case 1](image)
Case 2

The second test case involving a viscous flow over NACA0012 airfoil is at a low Mach number of 0.1, a Reynolds number of $10^5$ and at $3^\circ$ angle of attack. The solution is obtained using the second order flux computations with 0.2 CFL number. The computed pressure distribution around the airfoil is presented in Figure 8. The same solution is then obtained by applying V-cycle MG and Full MG algorithms. The convergence histories of the solutions in terms of the variation of the residual and the aerodynamic loads are given in Figure 9 and 10. The residual is defined as the L2 norm of the conservation of mass equation. The plots show similar trends in case 1, FMG and V-cycle MG solutions in Figure 9 exhibit about the same converge rate as expected and they are approximately 4 times faster than the single grid solution without the MG.

![Figure 8](image.png)  
**Figure 8** Pressure distribution in case 2

![Figure 9](image.png)  
**Figure 9** The residuals of fine grid level in case 2
A similar convergence rate is observed in the variation of drag, lift and pitching moment coefficients when the solutions with and without MG are compared as shown in Figure 10. Similar to the case 1, FMG and V-cycle solutions show a monotone convergence history.

**Figure 10** Variation of aerodynamic coefficients in case 2
The speed-up in the drag coefficient (CD) convergence in terms of CPU time (clock time) in fine grid level with and without MG is tabulated in Table 1. MG solutions include the overhead associated with grid coarsening procedure. Solution based on FMG and V-cycle MG is taken as a reference and CPU times to reach 20%, 10%, 5%, 1% and 0.1% error bands are recorded. The runtime needed to reach above error bands is documented for solution without MG. It is seen in Table 1 that the multigrid solutions have 20 to 30 fold faster convergence than the baseline solution.

<table>
<thead>
<tr>
<th>Error Band</th>
<th>Without MG</th>
<th>MG</th>
<th>Ratio</th>
</tr>
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<tbody>
<tr>
<td>20%</td>
<td>84,750</td>
<td>3,723</td>
<td>22.76</td>
</tr>
<tr>
<td>10%</td>
<td>215,860</td>
<td>6,934</td>
<td>31.13</td>
</tr>
<tr>
<td>5%</td>
<td>555,872</td>
<td>17,095</td>
<td>32.51</td>
</tr>
<tr>
<td>1%</td>
<td>No convergence</td>
<td>38,702</td>
<td>-</td>
</tr>
<tr>
<td>0.1%</td>
<td>No convergence</td>
<td>134,240</td>
<td>-</td>
</tr>
</tbody>
</table>

CONCLUSIONS

In this paper, we have presented a new grid coarsening technique based on placement of unstructured hybrid grid cells on a quadtree data structure. The algorithm developed and multigrid scheme is successfully implemented in an unstructured flow solver, SENSE2D. The multigrid solutions are obtained for inviscid and viscous flows over an airfoil. Numerical results show that the multigrid convergence rates are 5 to 30 fold greater than the baseline solution. The algorithm developed is currently being extended to three dimensional flows.

References