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BOUNDARY-LAYER STABILITY IN SUPERSONIC FLOWS

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ABSTRACT

A numerical analysis of the linear stability problem of supersonic flat plate boundary-layers is undertaken. The problem is of importance in the design of aerodynamic configurations such as orbital reentry vehicles and ICBMs. Perturbation equations are obtained for three-dimensional disturbance environment starting from the equations of motion for compressible flow, comprising momentum equations in Cartesian coordinates, energy equation and equation of state. With homogeneous boundary-conditions at the wall and the freestream, an eigenvalue problem results. The perturbation equations are solved for a two-dimensional flat plate boundary-layer using a Shooting Method with calorically perfect gas assumption. Unlike low-speed boundary-layers, where the most unstable wave is always two-dimensional according to Squire's theorem, in high-speed boundary-layers the most unstable wave is oblique for Mach numbers roughly above 2.0. In addition to the Tollmien-Schlichting mode of instability that is also present in low-speed boundary-layers, a second mode of instability occurs (Mack mode) whenever there is supersonic flow relative to the disturbance phase speed.

INTRODUCTION

Laminar-turbulent transition is an important phenomenon in fluid dynamics and aerodynamics with a large number of engineering applications. The reason is that it has very important effects on heat transfer and skin friction drag. Reduction of heating rates for the orbital reentry vehicles and ICBMs [Anderson, 1990], reduction of drag on the high subsonic-speed commercial aircraft wings are only few of areas that a good knowledge about transition is essential.

The Linear Stability Theory is mainly concerned with individual sine waves propagating in the boundary-layer parallel to the wall. These waves, referred to as the instability waves, were first explained by Rayleigh (1887) and Prandtl (1921) as small, regular oscillations traveling in the laminar boundary-layer. A complete theory of boundary-layer instability was studied by Tollmien (1929) and the total amplification of the most unstable frequencies was calculated by Schlichting (1933). That is why, the instability waves are also known as Tollmien-Schlichting waves. Besides, the first demonstrations of the existence of these waves were done by Schubauer and Skramstad (1948) through their well-known experiments. According to these, T-S waves are the first stage of the transition process.

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Compressibility makes this problem not only more realistic for most flow regimes, but also fundamentally more complex. The studies on this aspect start with Küchemann (1938), with an effort to try to build a compressible linear stability theory. Lees and Lin (1946) followed with theoretical investigations. It has been shown that a necessary condition for the existence of an unstable disturbance is:

$$\left[\frac{d}{dy}\left(\rho\frac{dU}{dy}\right)\right]_{y=y_s} = \mathbf{0}, \tag{1}$$

provided that $U(y_s) > U_\infty - a_\infty$, where ρ is the density, U the streamwise mean velocity component of the flow, y the normal distance from the wall and y_s is the location where the above equality is satisfied. This is the Generalized Inflection Point Theorem and is the extension of the well-known Inflection Point Theorem (or Rayleigh's Theorem) in incompressible flow.

Mack [Mack, 1984] outlined a complete numerical investigation for compressible laminar boundary-layers and discovered higher modes at supersonic speeds. Whenever the following condition holds:

$$\widetilde{M} = \frac{U - c_r}{a} > 1, \tag{2}$$

i.e. whenever there is a relative supersonic region in the flow, there exists an infinite number of unstable modes (or wave numbers). The first of these modes is related to the Tollmien–Schlichting mode in incompressible flow, but the higher or additional modes have no incompressible counterparts. The first of these additional modes has been referred to as the Mack mode. Unlike incompressible flow, where a two-dimensional disturbance is the most unstable at any Reynolds number according to Squire's theorem, for supersonic flow, the most unstable disturbance is always oblique [Özgen and Atalayer Kırcalı, 2008]. While the most unstable disturbances are oblique for the Tollmien-Schlichting mode of instability, the most unstable disturbances are two-dimensional for the Mack mode.

In this study, the linear stability analysis for a two-dimensional boundary-layer in zero pressure gradient is performed for planar (2-D) or oblique (3-D) waves. Calorically perfect gas and adiabatic wall assumptions are made.

THEORY

Mathematical Modeling

The mathematical model of the compressible stability problem starts with the 3-D Navier-Stokes equations for a compressible boundary-layer over an adiabatic flat plate. In this respect, the momentum, energy, continuity equations and the equation of state for a viscous, heat conducting, perfect gas in Cartesian coordinates are subject to method of small disturbances for linearization. Accordingly, each instantaneous flow property, i.e. velocity, pressure, temperature, density, viscosity and thermal conductivity is split into a steady mean (basic) and an unsteady fluctuating component [Özgen and Atalayer Kırcalı, 2008]:

$$\phi^*(x^*,y^*,z^*,t^*) = \Phi^*(x^*,y^*,z^*) + \hat{\phi}^*(x^*,y^*,z^*,t^*), \tag{3}$$

where $\boldsymbol{\phi}^*$ represents any one of $\boldsymbol{u}^*, \boldsymbol{v}^*, \boldsymbol{w}^*$ (velocity components in Cartesian coordinates); $\boldsymbol{p}^*, \boldsymbol{T}^*, \boldsymbol{\rho}^*$ (pressure, temperature and density); $\boldsymbol{\mu}^*$ (viscosity) and $\boldsymbol{k}^*$ (thermal conductivity). The following assumptions are made:

- Disturbances are small so quadratic or higher order terms involving perturbation quantities are neglected,
- Parallel flow assumption, i.e. $\boldsymbol{U}^* = \boldsymbol{U}^*(\boldsymbol{y}^*)$ and $\boldsymbol{W}^* = \boldsymbol{W}^*(\boldsymbol{y}^*)$ only and $\boldsymbol{V}^* = \mathbf{0}$ for the mean, basic flow.
- Temperature, $\boldsymbol{T}^*$ is a function of normal distance $\boldsymbol{y}^*$ only, i.e. $\boldsymbol{T}^* = \boldsymbol{T}^*(\boldsymbol{y}^*)$ and fluid properties, $\boldsymbol{\mu}^*, \boldsymbol{k}^*, \boldsymbol{C}_p^*, \boldsymbol{C}_v^*$ are functions of temperature only.

Velocities are non-dimensionalized by U_e^* (boundary-layer edge velocity) and lengths are made dimensionless by $L^* = \sqrt{\nu^* x^*/U_e^*}$ (Blasius length scale). Temperature, density, pressure, viscosity and heat conduction coefficient are non-dimensionalized by their respective freestream values, $T_e^*, \rho_e^*, p_e^*, \mu_e^*, k_e^*$. Therefore, Reynolds number is defined as $Re = \rho_e^* U_e^* L^*/\mu_e^*$.

The mean laminar flow is assumed to be influenced by a disturbance composed of a number of normal modes, which are propagating (traveling) waves of the form:

$$\hat{\phi}(x, y, z, t) = \bar{\phi}(y) e^{i(\alpha x + \beta z - \omega t)}. \tag{4}$$

In the above, α and β are x and z components of the wave number vector $\vec{k}$, and ω is the complex frequency defined as $\omega = \alpha c$ with $c = c_r + ic_i$ representing the complex wave velocity according to the temporal amplification formulation. The magnitude of the wave number vector is $k = \sqrt{\alpha^2 + \beta^2}$ and the wave angle is $\psi = \tan^{-1}(\beta/\alpha)$. The disturbance amplitude of the relevant variable is defined by $\bar{\phi}(y)$. The real part of the complex frequency $\omega_r = \alpha c_r$, is the circular frequency of the disturbance, while its imaginary part $\omega_i = \alpha c_i$, is the amplification rate. The imaginary part of the complex wave velocity c_i, is the amplification factor determining a stable $(c_i < 0)$, a neutrally stable $(c_i = 0)$, or an unstable $(c_i > 0)$ disturbance, while its real part c_r, is the phase velocity.

Substitution of the normal modes into the dimensionless, linearized system of equations and performing necessary algebra leads to the set of perturbation equations. The details of this procedure and the resulting equations are given in detail in [Malik and Anderson, 1991; Özgen and Atalayer Kırcalı, 2008].

The resulting equations are then transformed into a system of first order differential equations through the following variable definitions:

$$\begin{array}{llll} Z_1 = \alpha\bar{u} + \beta\bar{w} & Z_2 = Z_1' & Z_3 = \bar{v} & Z_4 = \bar{p}/\gamma M^2 \\ Z_5 = \bar{T} & Z_6 = Z_5' & Z_7 = \alpha\bar{w} - \beta\bar{u} & Z_8 = Z_7' \end{array} \tag{5}$$

where the Mach number is defined as $M = U_e^*/\sqrt{\gamma R^* T_e^*}$, γ being the ratio of specific heats and R^* being the gas constant. The system of equations can be expressed as:

$$Z_i' = \sum_{j=1}^{8} a_{ij} Z_j\,, \quad i = 1{,}8, \tag{6}$$

where, a_{ij} are the elements of the coefficient matrix and prime ($'$) denotes derivate with respect to y. The boundary conditions are:

$$Z_1(0) = Z_3(0) = Z_5(0) = Z_7(0) = 0 \quad (no\ slip), \tag{7}$$

$$Z_1, Z_3, Z_5, Z_7 \longrightarrow 0 \quad as \quad y \longrightarrow \infty \quad (freestream). \tag{8}$$

Two-dimensional basic flow equations are solved for velocity (U), temperature (T) and their derivatives with respect to y. These equations are:

For the velocity field:

$$2(\mu' U' + \mu U'') + FU' = 0, \tag{9}$$

For the temperature field:

$$2\left(\frac{\mu}{Pr} T'\right)' + FT' = -2(\gamma - 1) M^2 \mu (U')^2. \tag{10}$$

Prandtl number is defined as $Pr = \mu^* C_p^*/k^*$. Also notice that $\rho T = 1$ in the current formulation.

The boundary conditions are:

$$F(0) = F'(0) = 0, \ \ T'(0) = 0, \tag{11}$$

$$F' \longrightarrow 1, T \longrightarrow 1 \quad as \quad y \longrightarrow \infty. \tag{12}$$

Temperature dependent fluid properties are calculated using empirical formulae for example Sutherland's viscosity law [Malik and Anderson, 1991].

Solution Techniques

Equations (6) and the boundary conditions given in equations (7) and (8) constitute a characteristic value problem (i.e. eigenvalue problem) for the variables $(\alpha, \beta, \omega, Re)$. The problem is solved with the Shooting Method and Gram-Schmidt Orthonormalization defined in detail by [Özgen and Atalayer Kırcalı, 2008].

The stability diagrams, which are among the main goals of the study, are obtained using Newton Iteration in two variables. This method requires two initial points on the curve so that the iteration can proceed in the specified Reynolds number direction. These points can be found by using a function minimization routine employing the Simplex Method.

A FORTRAN code has been written by implementing the methods mentioned above, in order to solve the linear stability problem and hence, to obtain certain combinations of the eigenvalues $(\alpha, \beta, \omega, Re)$. The program is capable of solving the eigenvalue problem for either two-or-three dimensional sinusoidal disturbances using temporal amplification theory.

RESULTS AND DISCUSSION

Figure 1 represents the velocity and temperature profiles obtained numerically. The distribution of $(U'/T)'$ is presented in Figure 2, where $(U'/T)' = 0$ yields the generalized inflection point. It can be observed that, the generalized inflection point moves away from the wall as the Mach number increases, which is known to have a destabilizing effect.

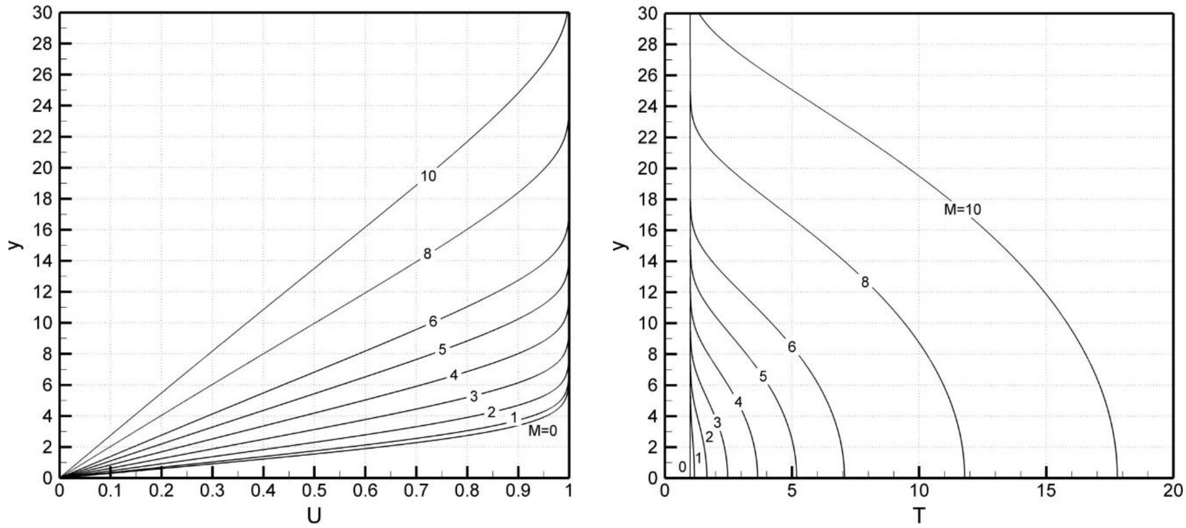


Figure 1: Velocity and temperature profiles for compressible flat-plate boundary-layers.

The existence of the Mack mode and the effect of wave three-dimensionality (wave angle) is depicted in Figure 3 for $M = 4$. In Figure 3a, the wave is purely two-dimensional ($\psi = 0°$) and both instability modes can be seen. Tollmien-Schlichting mode is observed at lower wave numbers, while the Mack mode is visible at the higher wave numbers. For this wave orientation, both modes have similar critical Reynolds numbers and amplification factors. However, when the wave angle is slightly increased to $20°$, the Mack mode is observed to undergo strong stabilization (reducing amplification factors and increasing critical Reynolds number) as shown in Figure 3b. When the wave angle is further increased to $60°$, the Mack mode no longer exists, meaning that it is totally stabilized, Figure 3c. Meanwhile, Tollmien-Schlichting mode continuously destabilizes (amplification factors increase, critical Reynolds numbers decrease) with increasing wave angle until $60°$, although the range of unstable wave numbers remain nearly constant. At the highest wave angle of $80°$, although the critical Reynolds number continues to decrease, the range of unstable wave numbers and amplification factors decrease, Figure 3d.

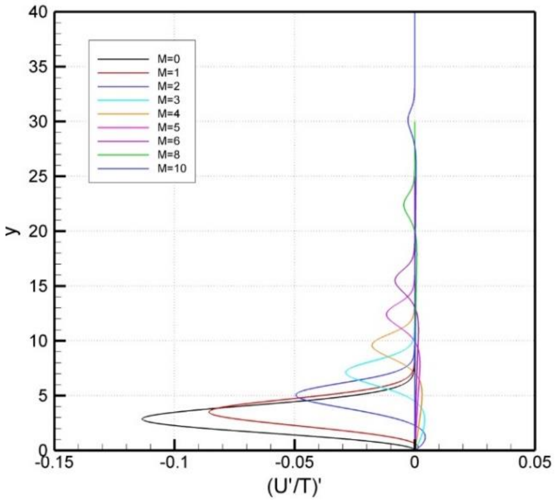
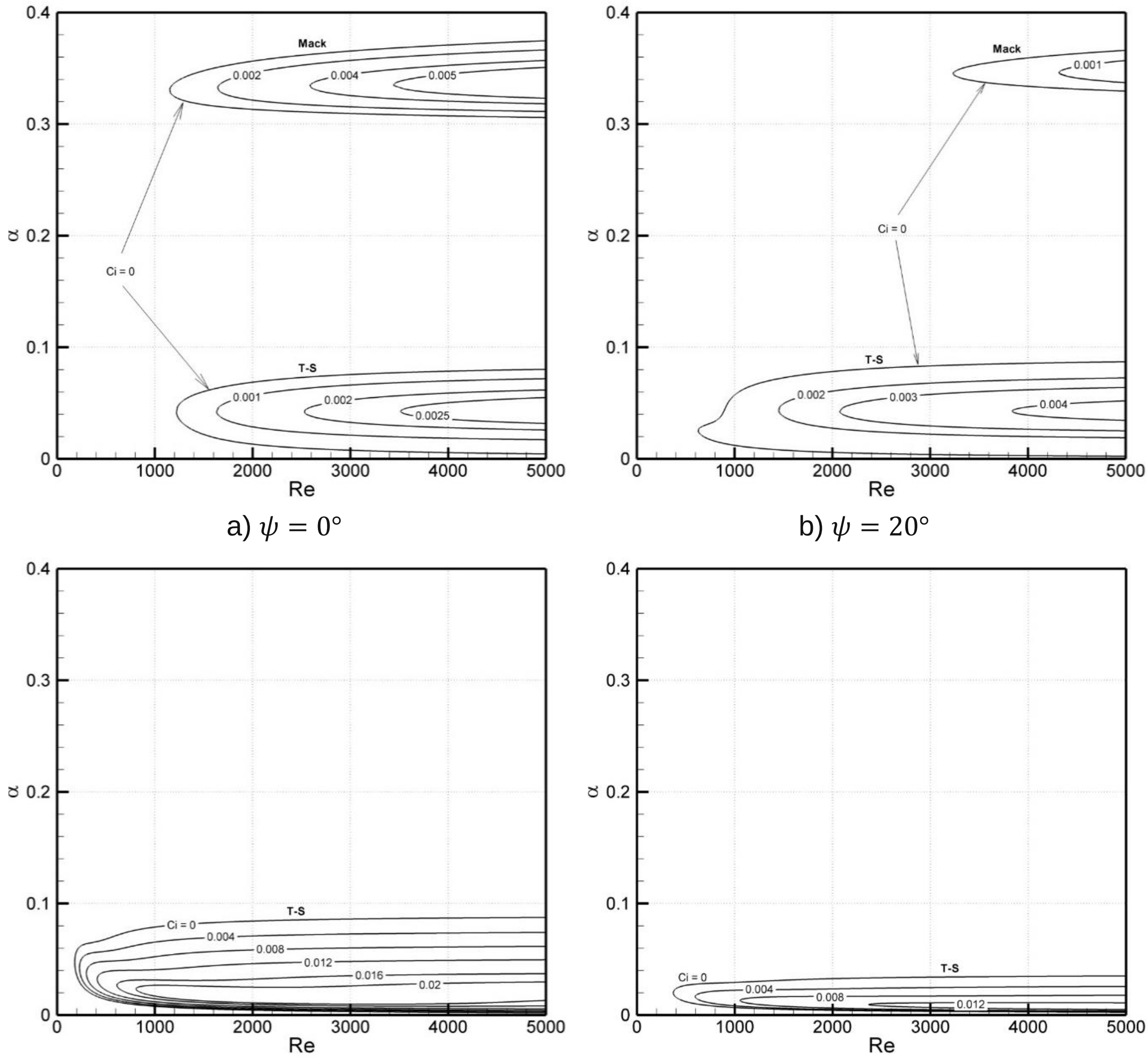


Figure 2: Second derivative of velocity and the generalized inflection point.



a) $\psi = 0°$ b) $\psi = 20°$

c) $\psi = 60°$ d) $\psi = 80°$

Figure 3: Variation of the stability curves with wave orientation for $\boldsymbol{M = 4}$.

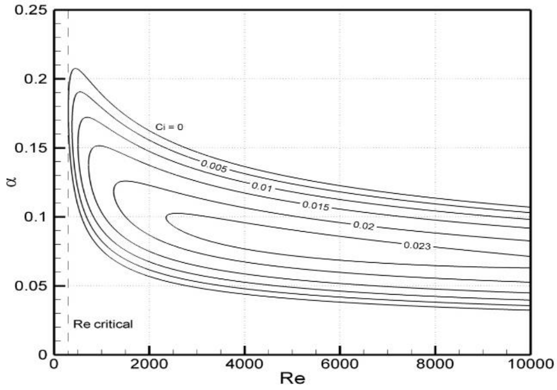
Figure 4 reveals several important phenomena. The figure depicts the stability diagrams for Mach numbers between 0 and 8, for the wave angles yielding the smallest critical Reynolds number, i.e. the most unstable wave orientation. All these curves correspond to the Tollmien-Schlichting mode because the Mack mode is rapidly stabilized even with very small wave orientation, as discussed above.

It is interesting to note that, while for $M \leq 1$ the stability diagrams show the characteristics of viscous instability (decreasing unstable wave number range with increasing Reynolds number), for $M > 2$, they demonstrate inviscid instability characteristics (range of unstable wave numbers remaining constant with increasing Reynolds number). This is due to the presence of the generalized inflection point, which results in stronger instability according to the Generalized Inflection Point Theorem.

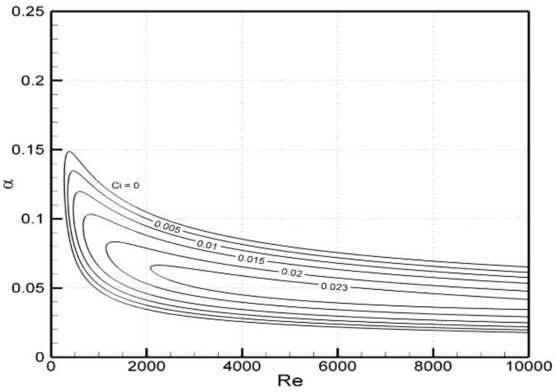
Another interesting observation is that for $M > 2$, the most unstable wave direction remains almost constant at around $\Psi = 60^{\circ}$. Together with this, for $M > 4$, stability diagrams look very similar to each other from many aspects: critical Reynolds numbers, range of unstable wave numbers and amplification factors. For these Mach numbers, the unstable wave number range is roughly $0 < \alpha < 0.1$. The critical Reynolds numbers are around $150 - 160$ (see also Figure 5) and the maximum amplification factors are in the vicinity of $c_i = 0.02$. These observations suggest that the stability characteristics become almost independent of Mach number for $M > 4$ for three-dimensional waves, which are shown to be the most unstable waves, in contrast with Squire's Theorem in incompressible stability.

Figure 5, summarizes the observations made in Figure 4, in terms of the critical Reynolds number and the wave angle.

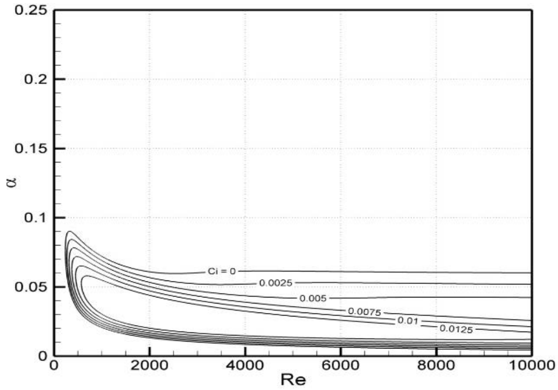
Figure 6 shows the comparison of the neutral stability curves obtained for two Mach numbers with literature data [Pinna, 2012]. Although very different numerical methods are used in both studies ([Pinna, 2012] used a Chebyshev Collocation Method, while Shooting Method is used in the present study), the results agree remarkably well. The critical Reynolds numbers and the range of unstable wave numbers are in very good agreement. It can also be observed that both studies capture the Tollmien-Schlichting mode and the Mack mode at $M = 4$.



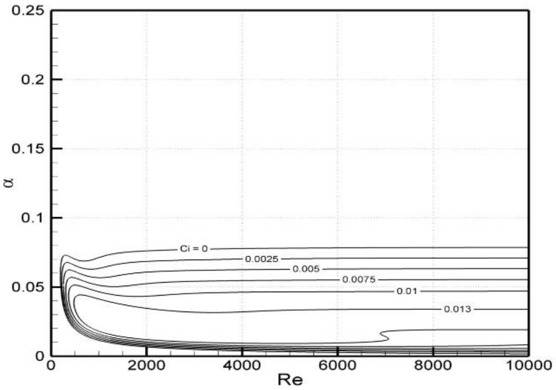
a) $M = 0, \psi = 0°$



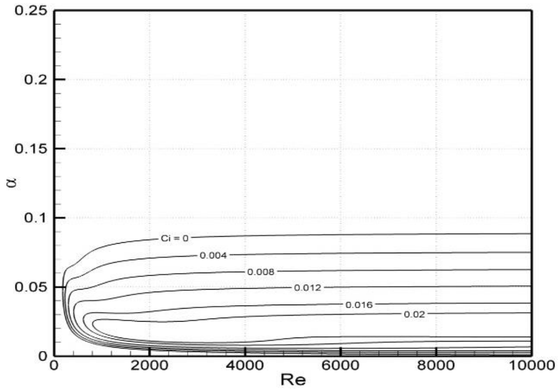
b) $M = 1, \psi = 0°$



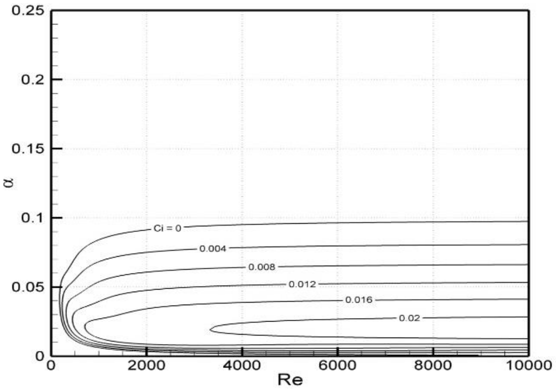
c) $M = 2, \psi = 45°$



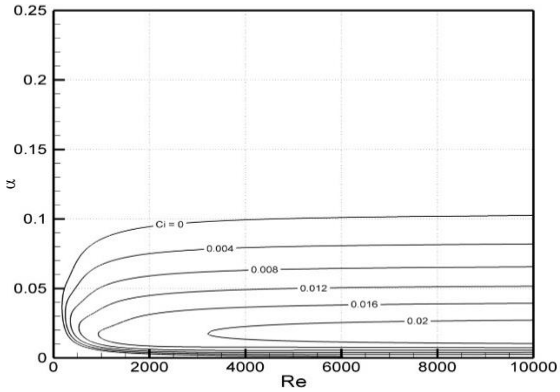
d) $M = 3, \psi = 55°$



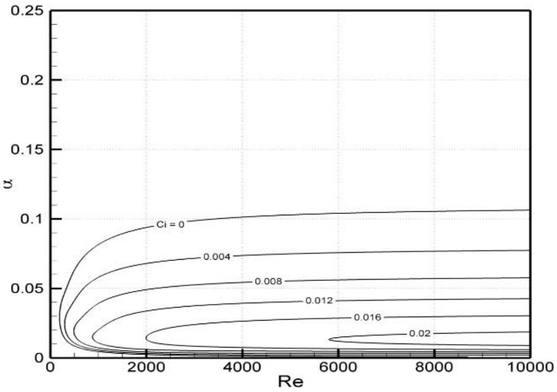
e) $M = 4, \psi = 60°$



f) $M = 5, \psi = 60°$



g) $M = 6, \psi = 60°$



h) $M = 8, \psi = 60°$

Figure 4: Stability curves for the most unstable wave directions.

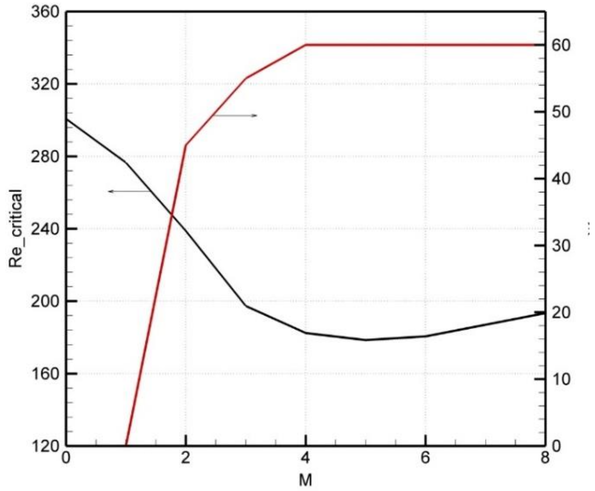
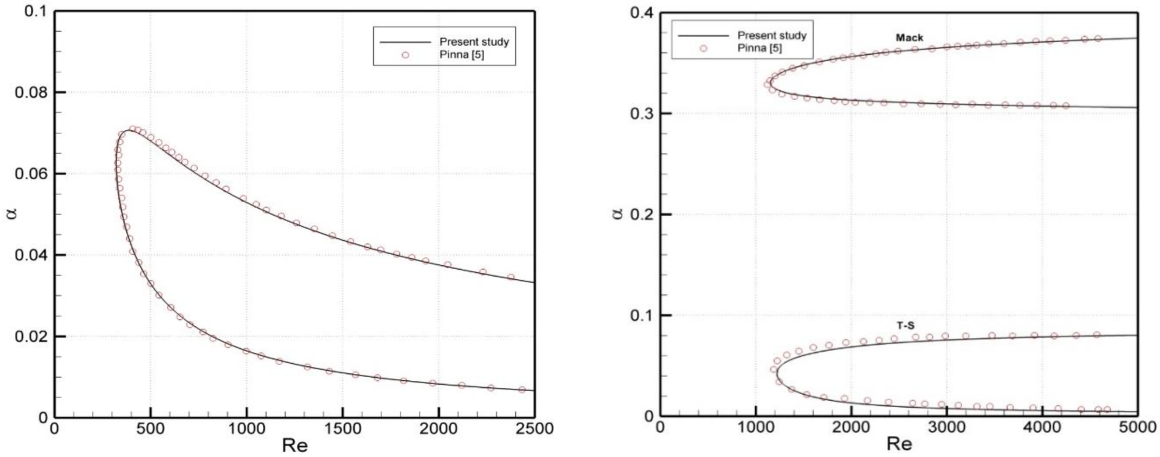


Figure 5: Variation of critical Reynolds number and the most unstable wave angle with Mach number.



a) $M = 2, \psi = 0°$ b) $M = 4, \psi = 0°$

Figure 6: Comparison of neutral stability curves for two-dimensional disturbances with literature data [5].

CONCLUSIONS

Stability characteristics of compressible flat-plate boundary-layers are determined using the Linear Stability Theory. The results confirm that as soon as there is relative supersonic flow, a second mode of instability (Mack mode) is observed in addition to the usual Tollmien-Schlichting mode. The Mack mode is rapidly stabilized as wave angle is increased and only the Tollmien-Schlichting mode is observed for higher wave angles. The most unstable wave directions are typically around $\Psi = 60^{\circ}$ for moderate and high Mach numbers.

The most interesting and important result of this study is the behavior of the stability curves for $M > 4$ for oblique waves. The stability characteristics become almost independent of Mach number for $M > 4$ for three-dimensional waves (oblique waves), which are also shown to be the most unstable waves for these Mach numbers.

A natural next step would be to include the real gas effects in the analysis with more fidelity like in [Malik and Anderson, 1991], where real gas effects have been considered with thermal and chemical equilibrium assumption.

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