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ORBIT DETERMINATION FROM DOPPLER AND ANGLE DATA

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ABSTRACT

In this work, we investigate how the Laplace method, which is an angle-only preliminary orbit determination method, is modified if the range-rate data from Doppler measurements are associated with the angle-only data. The Laplace method requires three angular observations at three different time instants and we add three range-rate data at these instants to the angular data. With the use of this additional data, we modify the Laplace's method such that the state vector of the intermediate instant can be obtained from a simple analytical formula. Then, we compare the developed method with the Laplace method, Gauss method, and Escobal's method which is also a modification of Laplace's method for these mixed data.

INTRODUCTION

Long before that the space race of the twentieth century led the launch of first artificial satellite had humanity started observing the night sky and developing methods to determine the orbits of heavenly bodies. With the efforts of establishing astronomical tables for planets and stars, numerous methods of orbit determination have been developed for centuries. Those methods flourished with the application of them on thousands of space missions for about sixty years.

Based on the optical observations of astronomical objects, the angle-only methods are one of the most well-established (preliminary) orbit determination methods as they have been applied for centuries. Among these methods, we consider the Laplace's method (see [Klokacheva, 1991; Vallado, 2007]) and Gauss's method [Curtis, 2013]. Although these methods are really well-established, with the advent of numerical techniques, there are variations of them which are better suited with numerical approaches. For example, in [Gooding, 1990], Gooding proposed an approach which turns three angular data problem handled with Laplace or Gauss methods to the Lambert's problem which is another well-established method requiring two geocentric position vector measurements (see [Curtis, 2013] for the standard description of the problem).

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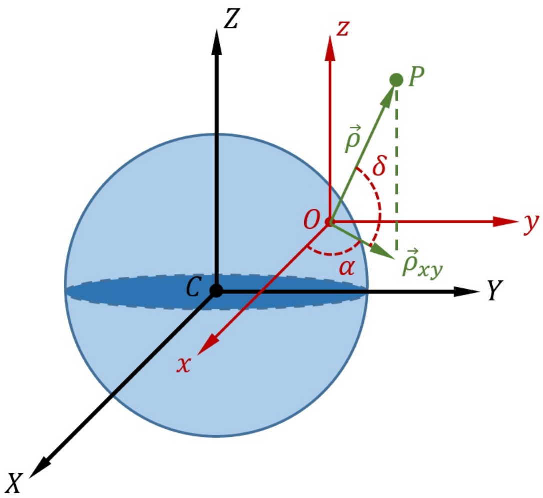

Figure 1: The direction of the observed object P relative to the observer O is given by the right ascension α and declination δ angles in the topocentric equatorial frame xyz.

In this work, we investigate how the angle-only orbit determination methods are modified if the range rate data from Doppler measurements are associated with the angle-only data. To do this, we first consider that the range-rate data are added without changing the problem description and the method is updated with the use of the added data. This is a standard mixed data orbit determination problem which is solved by a numerical modification to Laplace method in [Escobal, 1965] (see also [Yanez,Mercier and Dolado, 2017] for a recent numerical approach). Here, we consider the problem analytically which is in the same spirit as the Gibbs method (see [Curtis, 2013] for example) which is basically an extension of Lambert's problem with one more position vector data. Afterwards, we are planning to change the problem definition such that reducing the amount of angle-only data requirements of the standard angle-only methods and adding the range-rate data in place of them.

METHOD

Angular observation of an astronomical object or a satellite by an earth-based observer is basically determining two angular quantities representing the direction of the observed object relative to the observer. Since this observation is made by an earth-based observer, the angles are measured in a topocentric frame origined at the position of the observer. This topocentric frame can be chosen as the topocentric equatorial frame or the topocentric horizon frame. As we ultimately have the position vectors in the geocentric equatorial frame, it would be better to use the topocentric equatorial frame and represent the relative direction with the use of topocentric right ascension α and declination δ angles as seen in Fig. 1. The direction of the relative position vector $\vec{\rho}$ is given by the unit vector $\hat{\rho}$ which can be calculated in terms of α and δ as

$$\hat{\rho} = \cos\delta\cos\alpha\hat{I} + \cos\delta\sin\alpha\hat{J} + \sin\delta\hat{K}. \tag{1}$$

The magnitude of the relative position vector $\vec{\rho}$ is called the slant range and denoted as ρ.

For the both Gauss's method and Laplace's method, three observations at times t_1 , t_2 and t_3 are required. The data for these three observations are the right ascension α and the declination δ angles which are measured through optical sightings. The measurements could be from the same observation site or different observation sites. The geocentric position vector $\vec{r}$, that is the inertial position of the object, can be written in terms of the observer's geocentric position vector $\vec{R}$, the slant range ρ , and the topocentric direction vector (or the line of sight vector) $\hat{\rho}$ as

$$\vec{r}_i = \vec{R}_i + \rho_i\hat{\rho}_i, \tag{2}$$

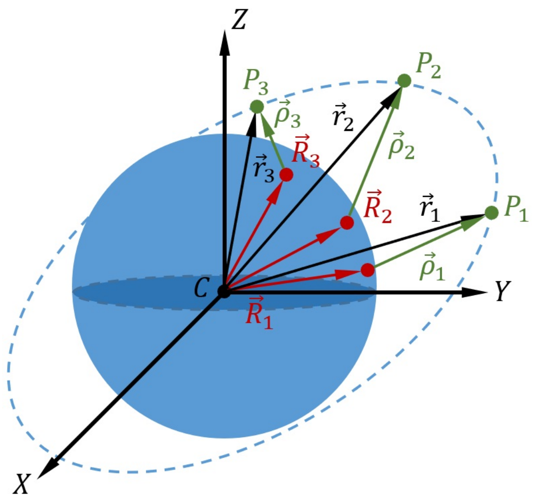

Figure 2: For three observations, the geocentric position vector $\vec{r}$ of the satellite and the relative position vector $\vec{\rho}$ of the satellite with respect to observeration site given with the geocentric position vector $\vec{R}$.

where i represents the three observation times as $i = 1, 2, 3$, see Fig. 2. In these equations, $\vec{R}_i$'s are the known geocentric position vectors for the observation sites once the time of observation is known. The topocentric direction vectors $\hat{\rho}_i$'s are to be found from (1). Thus, the slant ranges ρ_i's, and then in turn, the geocentric position vectors $\vec{r}_i$'s are the unknowns. To determine the orbit, one needs to determine the state vector at an instant.

Now, let us discuss the Laplace's method and the Gauss's method, respectively. Since the main result for this abstract is obtained for the Laplace's method, it is discussed a bit more in detail.

Laplace Method

For this part, we followed the exposition of the Laplace's method given in [Vallado, 2007]. In this method, to use the equation of motion for the two-body problem, that is $\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r}$, the first and second derivatives of

$$\vec{r}\,(t) = \vec{R}\,(t) + \rho\,(t)\,\hat{\rho}\,(t)\,, \tag{3}$$

are taken to have

$$\vec{v} = \dot{\vec{R}} + \dot{\rho}\hat{\rho} + \rho\dot{\hat{\rho}}, \tag{4}$$

$$\vec{a} = \ddot{\vec{R}} + \ddot{\rho}\hat{\rho} + 2\dot{\rho}\dot{\hat{\rho}} + \rho\ddot{\hat{\rho}}. \tag{5}$$

Combining the last equation with $\vec{a} = -\frac{\mu}{r^3}\vec{r}$ and rearranging gives

$$\ddot{\rho}\hat{\rho} + 2\dot{\rho}\dot{\hat{\rho}} + \rho\left(\ddot{\hat{\rho}} + \frac{\mu}{r^3}\hat{\rho}\right) = -\ddot{\vec{R}} - \frac{\mu}{r^3}\vec{R}. \tag{6}$$

This equation is considered at the middle instant. The rate of change of the position vector of the observation site is

$$\dot{\vec{R}} = \vec{\Omega} \times \vec{R}, \tag{7}$$

where $\vec{\Omega}$ is the angular velocity of Earth. In addition, the accelaration of the observation site is

$$\ddot{\vec{R}} = \vec{\Omega} \times \dot{\vec{R}}. \tag{8}$$

Each $\hat{\rho}$'s is at a particular time, hence we can use the Lagrange interpolation formula to derive an expression for $\hat{\rho}\left(t\right)$ in the time interval $[t_1, t_3]$ and its derivatives in this interval as

$$\hat{\rho}\left(t\right)=\frac{\left(t-t_2\right)\left(t-t_3\right)}{\left(t_1-t_2\right)\left(t_1-t_3\right)}\hat{\rho}_1+\frac{\left(t-t_1\right)\left(t-t_3\right)}{\left(t_2-t_1\right)\left(t_2-t_3\right)}\hat{\rho}_2+\frac{\left(t-t_1\right)\left(t-t_2\right)}{\left(t_3-t_1\right)\left(t_3-t_2\right)}\hat{\rho}_3, \tag{9}$$

$$\dot{\hat{\rho}}\left(t\right)=\frac{2t-t_2-t_3}{\left(t_1-t_2\right)\left(t_1-t_3\right)}\hat{\rho}_1+\frac{2t-t_1-t_3}{\left(t_2-t_1\right)\left(t_2-t_3\right)}\hat{\rho}_2+\frac{2t-t_1-t_2}{\left(t_3-t_1\right)\left(t_3-t_2\right)}\hat{\rho}_3, \tag{10}$$

$$\ddot{\hat{\rho}}\left(t\right)=\frac{2}{\left(t_1-t_2\right)\left(t_1-t_3\right)}\hat{\rho}_1+\frac{2}{\left(t_2-t_1\right)\left(t_2-t_3\right)}\hat{\rho}_2+\frac{2}{\left(t_3-t_1\right)\left(t_3-t_2\right)}\hat{\rho}_3. \tag{11}$$

Then, assuming the magnitude of the position vector, that is

$$r=\sqrt{R^2+2\rho\vec{R}\cdot\hat{\rho}+\rho^2}, \tag{12}$$

is known, we can write (6) in matrix form as

$$\left[\begin{array}{c|c|c} \hat{\rho} & 2\dot{\hat{\rho}} & \ddot{\hat{\rho}}+\frac{\mu}{r^3}\hat{\rho} \end{array}\right]\left[\begin{array}{c} \ddot{\rho} \\ \dot{\rho} \\ \rho \end{array}\right]=-\left[\ddot{\vec{R}}+\frac{\mu}{r^3}\vec{R}\right], \tag{13}$$

which can be solved for ρ as

$$\rho=-\frac{2D_1}{D}-\frac{2\mu D_2}{r^3 D}, \tag{14}$$

after applying the Cramer's rule and using some matrix algebra. Here, the determinants D, D_1, and D_2 are defined as

$$D\equiv 2\left|\begin{array}{c|c|c} \hat{\rho} & \dot{\hat{\rho}} & \ddot{\hat{\rho}} \end{array}\right|, \tag{15}$$

$$D_1\equiv 2\left|\begin{array}{c|c|c} \hat{\rho} & \dot{\hat{\rho}} & \ddot{\vec{R}} \end{array}\right|, \tag{16}$$

$$D_2\equiv 2\left|\begin{array}{c|c|c} \hat{\rho} & \dot{\hat{\rho}} & \vec{R} \end{array}\right|. \tag{17}$$

One way to solve (14) is to assume a value for ρ, and then iterating to obtain the final value. Or, substituing (14) in (12) yields

$$r^8+\left(\frac{4CD_1}{D}-\frac{4D_1^2}{D^2}-R^2\right)r^6+\mu\left(\frac{4CD_2}{D}-\frac{8D_1D_2}{D^2}\right)r^3-\frac{4\mu^2 D_2}{D^2}=0, \tag{18}$$

where $C=\hat{\rho}\cdot\overrightarrow{R}$. In the case where multiple real roots exist, each root must be tested. Once the correct r value is found, the final step is to find ρ through (14), and then, using ρ value can be used to find the geocentric position vector for the middle instant.
The process is then repeated for $\dot{\rho}$ to have the solution

$$\dot{\rho}=-\frac{D_3}{D}-\frac{\mu D_4}{r^3 D}, \tag{19}$$

where D_3 and D_4 are found as

$$D_3\equiv\left|\begin{array}{c|c|c} \hat{\rho} & \ddot{\vec{R}} & \ddot{\hat{\rho}} \end{array}\right|, \tag{20}$$

$$D_4\equiv\left|\begin{array}{c|c|c} \hat{\rho} & \vec{R} & \ddot{\hat{\rho}} \end{array}\right|. \tag{21}$$

Using ρ and $\dot{\rho}$ values in (4) yields the velocity vector for the middle instant. Thus, the state vector for the middle instant is determined.

Gauss Method

Due to the conservation of the angular momentum, the geocentric position vectors $\vec{r}_i$'s lie in the same plane, which is the orbital plane. Then, for noncollinear geocentric position vectors, one of the $\vec{r}_i$ vectors can be written as a linear combination of the other two as

$$\vec{r}_2 = c_1\vec{r}_1 + c_3\vec{r}_3. \tag{22}$$

In addition, using the Lagrange coefficients, the $\vec{r}_1$ and $\vec{r}_3$ vectors can be written in terms of the state vector in the middle time instant as

$$\vec{r}_1 = f_1\vec{r}_2 + g_1\vec{v}_2, \tag{23}$$
$$\vec{r}_3 = f_3\vec{r}_2 + g_3\vec{v}_2. \tag{24}$$

Using these three vector equations, first it is possible to write the c_1 and c_3 coefficients in terms of the Lagrange coefficients. Then, expanding the Lagrange coefficients in $\tau_1 = t_2 - t_1$ and $\tau_3 = t_3 - t_2$ time intervals and extensive use of the vector algebra yield the state vector in the middle time point. For a detailed discussion of the Gauss's method see [Curtis, 2013].

Escobal's Modification of Laplace Method

One of the methods suggested in the literature that assumes the range-rate to be known is presented by Escobal in [Escobal, 1965]. In this section, we recapitulate this discussion.

The method starts to deviate from Laplace method by taking the dot product of (6) with $\hat{\rho}$ yielding

$$-\frac{\mu}{r^3}\rho - \frac{\mu}{r^3}\hat{\rho}\cdot\vec{R} = \ddot{\rho} + \rho\hat{\rho}\cdot\ddot{\hat{\rho}} + \hat{\rho}\cdot\ddot{\vec{R}}. \tag{25}$$

Then, using $\hat{\rho}\cdot\dot{\hat{\rho}} = 0$ and its time derivative

$$\hat{\rho}\cdot\ddot{\hat{\rho}} = -\dot{\hat{\rho}}\cdot\dot{\hat{\rho}}, \tag{26}$$

one can write (25) as,

$$\left(\dot{\hat{\rho}}\cdot\dot{\hat{\rho}} - \frac{\mu}{r^3}\right)\rho = \ddot{\rho} + \hat{\rho}\cdot\ddot{\vec{R}} + \frac{\mu}{r^3}\hat{\rho}\cdot\vec{R}. \tag{27}$$

Here, note that $\ddot{\hat{\rho}}$, which is one of the terms determined from the interpolating function (9), is eliminated from this expression. Then, (27) can be rearranged to have the range at the second time point as

$$\rho = \frac{A + B/r^3}{C + D/r^3}, \tag{28}$$

where A, B, C, and D are calculated at the second time point as

$$A \equiv \ddot{\rho} + \hat{\rho}\cdot\ddot{\vec{R}}, \qquad B \equiv \mu\hat{\rho}\cdot\vec{R}, \qquad C \equiv \dot{\hat{\rho}}\cdot\dot{\hat{\rho}}, \qquad D \equiv -\mu. \tag{29}$$

As in the Laplace's method, one way to solve (28) is to assume a value for ρ, and then iterating to obtain the final value. Or, substituing (28) in (12) yields again an 8th order polynomial in r which can again be solved by Newton-Raphson like iterative methods. Once r and ρ are obtained, the state vector is obtained from (3) and (4).

APPLICATION

Let us first consider how the Laplace's method is modified with the inclusion of Doppler data yielding $\dot{\rho}$ values at each instant. Considering (6) for the middle instant, the $\dot{\rho}$ value is now known and to

1 25544U	98067A	19241.83275787	.00001839	00000-0	39700-4	0	9998
2 25544	51.6448	355.9501	0007912	342.5346	99.5642	15.50401679186686		

Table 1: TLE for 29th of August 2019, 19:59:10 UTC

determine $\ddot{\rho}$ value following the spirit of the Laplace's method, we can have the interpolating function for $\dot{\rho}$ as

$$\dot{\rho}(t) = \frac{(t-t_2)(t-t_3)}{(t_1-t_2)(t_1-t_3)}\dot{\rho}_1 + \frac{(t-t_1)(t-t_3)}{(t_2-t_1)(t_2-t_3)}\dot{\rho}_2 + \frac{(t-t_1)(t-t_2)}{(t_3-t_1)(t_3-t_2)}\dot{\rho}_3. \tag{30}$$

From this interpolating function one can have $\ddot{\rho}(t)$ as

$$\ddot{\rho}(t) = \frac{(2t-t_2-t_3)}{(t_1-t_2)(t_1-t_3)}\dot{\rho}_1 + \frac{(2t-t_1-t_3)}{(t_2-t_1)(t_2-t_3)}\dot{\rho}_2 + \frac{(2t-t_1-t_2)}{(t_3-t_1)(t_3-t_2)}\dot{\rho}_3. \tag{31}$$

Putting the $\ddot{\rho}(t=t_2)$ value in (6) and taking the dot product with $\vec{R}\times\hat{\rho}$ yields

$$2\dot{\rho}\left(\vec{R}\times\hat{\rho}\right)\cdot\dot{\hat{\rho}} + \rho\left(\vec{R}\times\hat{\rho}\right)\cdot\ddot{\hat{\rho}} = -\left(\vec{R}\times\hat{\rho}\right)\cdot\ddot{\vec{R}}. \tag{32}$$

This equation can easily be solved for ρ as

$$\rho = -\frac{\left(\vec{R}\times\hat{\rho}\right)\cdot\ddot{\vec{R}} + 2\dot{\rho}\left(\vec{R}\times\hat{\rho}\right)\cdot\dot{\hat{\rho}}}{\left(\vec{R}\times\hat{\rho}\right)\cdot\ddot{\hat{\rho}}}. \tag{33}$$

Remember that $\hat{\rho}$ and $\dot{\rho}$ are obtained through observations. The vector $\vec{R}$ is the known geocentric position vector of the observation site, and the inertial acceleration of the observation site can easily be calculated from $\vec{R}$ and the angular velocity of Earth. The remaining quantities on the right hand side are the $\dot{\hat{\rho}}$ and $\ddot{\hat{\rho}}$ vectors which can be calculated by use of the Lagrange interpolation formula involving three angular observations. Thus, ρ is directly calculable from (33).

RESULTS AND DISCUSSION

The main result of this paper is the solution for ρ given in (33). After obtaining ρ value, $\vec{r}$ and $\vec{v}$ can be found from (2) and (4), respectively; and in turn, the state vector $(\vec{r}_2, \vec{v}_2)$ becomes determined.

Numerical Results

In this study, to understand how having an analytical solution for the range effects the accuracy in preliminary orbit determination, we developed the codes for the Laplace method, the Escobal's modification of Laplace method, and the modification that we presented. In addition, for the Gauss's method, we used the code developed by Curtis [Curtis, 2013]. Since we have not had an observation based data for the angles and the range-rates, we used the two-line element (TLE) data from Space-Track.org to generate the input for the developed codes via a two-body propagator code which also includes the J2 perturbations averaged over a period. We studied the visible pass of International Space Station (ISS) over Ankara on 30th of August 2019 between 05:02:43 and 05:07:27 in the local time (UTC+03:00). To generate the data, we used the TLE given in Table 1, and obtain the angle and range-rate data given in Table 2. Using the input data in Table 2, the four preliminary orbit determination methods mentioned above provide the classical orbital element (COE) values at the second time point as given in Table 3 in which the propagated COE values are also presented. In addition, the state vector output of these methods for the second time point are also given in Table 4 again together with the propagated state vector.

The results in the Table 3 and Table 4 shows that the Gauss method is far more accurate than the Laplace method and the modifications of Laplace method with the range-rates. In addition,

Instant (UTC+03:00)	α	δ	$\dot{\rho}$ (m/s)
05:04:17	355.780°	64.0511°	3.46074×10^{-3}
05:05:17	135.154°	72.1947°	4.69583
05:06:17	151.300°	52.7665°	6.29005

Table 2: The angular (topocentric right-ascension and declination) and range-rate data for the visible pass of ISS over Ankara on 30th of August 2019 starting and ending at the local times (UTC+03:00) of 05:02:43 and 05:07:27, respectively.

	e	Ω	i	ω	θ	a (km)
Propagated	7.91200×10^{-4}	354.690°	51.6448°	343.474°	1.37510°	6793.7
Laplace	0.244440	354.813°	52.0371°	56.2401°	6.37031°	9051.7
Gauss	3.21672×10^{-3}	354.708°	51.6563°	47.8353°	14.4197°	6814.4
Escobal	0.465945	356.066°	52.7169°	52.0675°	9.98270°	12809
Developed	0.668629	354.754°	52.4888°	58.6413°	4.60440°	20877

Table 3: Comparison of COEs generated by the four preliminary orbit determination methods with the COEs obtained by the two-body propagator in which average J2 effects are added.

	$\vec{r}$	$\vec{v}$
Propagated	$\left(3493.2, \ 3422.1, \ 4714.5 \right)$	$\left(-6.5436, \ 2.8337, \ 2.8002 \right)$
Laplace	$\left(3475.3, \ 3439.9, \ 4793.4 \right)$	$\left(-7.2098, \ 3.1733, \ 3.2151 \right)$
Gauss	$\left(3493.0, \ 3422.3, \ 4715.4 \right)$	$\left(-6.5535, \ 2.8356, \ 2.8054 \right)$
Escobal	$\left(3466.7, \ 3448.5, \ 4831.4 \right)$	$\left(-7.6697, \ 3.4181, \ 3.7879 \right)$
Developed	$\left(3449.5, \ 3465.5, \ 4906.4 \right)$	$\left(-8.3052, \ 3.6302, \ 3.7202 \right)$

Table 4: Comparison of the state vector $(\vec{r}, \vec{v})$ generated for the four preliminary orbit determination methods with the $(\vec{r}, \vec{v})$ obtained by the two-body propagator in which average J2 effects are added.

the Laplace method provided more accurate results compared to its modifications. Finally, the modification of the Laplace method given by Escobal provided more accurate results compared to the method we developed. These conclusions are derived from only one pass of a specific satellite for a specific set of data in this pass. With only these results, these conclusions about the comparison of the methods cannot be definitive. One definitely needs to study more passes of the same satellite with different topocentric elevations. In addition, the same pass should be studied with different time intervals, with various equal elevation changes, with the symmetrized data around the highest elevation point. Furthermore, several other satellites should be studied in various orbits such as GEO, MEO, sun-synchronous, and highly elliptic (for example, Molniya).

CONCLUSIONS

In this study, we obtained an analytical expression for the solution of the mixed data problem in the case of Laplace method. We studied this solution together with other preliminary orbit determination methods. We observed that providing an analytical solution for the mixed data problem does not provide an increased accuracy in preliminary orbit determination. In addition, the case we studied indicated that adding more input data in the form of range-rate also does not increase the orbit determination accuracy. These observations are not conclusive as we have not studied several different data sets and such a study would be the next step in this work.

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