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UNCERTAIN BUCKLING OF COMPOSITE COLUMNS BY CONVEX MODELING

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ABSTRACT

Until now, deterministic approaches have been applied in many studies on buckling of columns. Since the properties and dimensions of the material may vary due to the defects in production stages, which have a significant effect on the critical buckling load, should be considered as uncertain. In this study, closed form linear buckling formulas including the material and dimensional uncertainties have been developed by using the First-Order Shear Deformation Theory for orthotropic composite columns with I and L sections. The lowest critical buckling equations for different uncertainty levels are obtained analytically by using convex model. Resulting equations are then solved by the Lagrange multipliers method. For the verification of the equations, columns are also modeled by finite element method. It's seen that, even minor deviations on the parameters lead to remarkable losses on critical buckling loads.

INTRODUCTION

Composite columns are being widely used in industry because they offer high strength-to-weight and high stiffness to weight ratios. Because of advanced properties, composite materials are getting more attention in aerospace industry. Therefore, it is crucial to know the real buckling behavior of these structures to minimize the life and property losses. For this purpose, the uncertainty phenomena [Oktem and Adali, 2018] should be included in the analysis to get more realistic results. Buckling occurs when a member is under compressive load, particularly in thin-walled structures. When the applied load reaches a point that unbalances the column, large lateral deviations occur on the structure and this phenomenon is defined as buckling. The critical buckling load is an important limit for the structural stability of the column. Engineers should be able to predict whether the columns are structurally stable under a certain axial load. Therefore, critical buckling load has an important role in columns.

In many analytical studies that have investigated the buckling of columns, certain (deterministic) approaches have been applied. Barbero and Tomblin (1993), based on the Euler-Bernoulli beam theory, obtained the buckling load equation in the I-section composite

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columns with the classical lamination theory. However, in analytical results that were compared experimentally, it was found that the study could be valid only for long columns. Akbulut et al. (2010) examined the behavior of buckling based on the first-order shear deformation theory (FSDT) in the columns of gradually changed cross-section. In the region where the column section varies gradually, it was observed that the buckling resistance is increased in case of using the curved structure. The analytically obtained results were validated by finite element method. Schnabl and Planinc (2011) presented an effective mathematical model to study the geometrically perfect elastic two-layer composite columns with the sliding behavior of layers. The analytical model was based on the linearized stability theory and could predict the exact critical buckling loads. According to the reported information, the negative impact of transverse shear deformation on buckling can be up to 20% for wood composite columns and up to 40% for composite columns that can exhibit very flexible behavior. This work showed that the utilization of the first-order shear deformation theory in composite columns is important. However, since the results obtained in all reported work are found by considering certain (deterministic) approaches, they cannot give a healthy information about the real buckling behavior of the columns as a result of fluctuations in material properties and dimensions.

In the literature, other than columns, there are studies related to buckling analysis considering the material properties, dimensions and applied load of the shell and conical elements as uncertain [Zhang & Ellingwood, 1995; Papadopoulos & Papadrakakis, 2005; Gamez-Montero et al., 2009; Ifayefunmi and Błachut, 2011; Tomás and Tovar, 2012]. These studies, in general, reveal that the issue of uncertainty is very important for structures. Morgan et al. (1965), in the experimental study on elastic stability of cylindrical and conical shell structures, observed that there are differences between the theoretical results and experimental results. They have revealed that this was due to defects in the material. Therefore, the use of uncertainty methods in the analysis of buckling in structures is taking a crucial role to obtain more realistic results.

Convex model, probabilistic and statistical approaches are used to examine the uncertainty for the structures. Probabilistic calculations and statistical approaches require probabilistic density function and large amounts of experimental data as well. The advantage of the convex model, which can give more accurate results to the users in the relevant field, is that it can be used in uncertainty studies without the need of experimentation [Ben-Haim ve Elishakoff, 1990; Qui, 2005; Radebe ve Adali, 2014; Oktem ve Adali, 2018]. For this reason, the convex model is used in this study.

Except for the convex model, some uncertainty applications are used based on probabilistic calculations. Likewise, studies on imperfect beams using convex model may also found in the literature [Verhaeghe et al., 2013]. Herein, cosinusoidal polynomials were used to obtain certain (deterministic) buckling loads. Then, probabilistic calculations and convex model were utilized for uncertainty analysis. A significant similarity was observed between probabilistic calculations and convex models. Qui et al. (2006) examined the nonlinear buckling of a column. In cases where the deflections in the center of the columns are different from zero, minimum buckling load is determined by using first and second order Taylor series expansions. While the random coefficients of the Fourier series were used in the model based on probabilistic estimation, the upper-lower limit coefficients were used in the convex model. Zhang et al. (2017) examined the geometry of a thin imperfect plate by the method of uncertainty. Trigonometric series were used to determine the defects and Monte Carlo simulation was used to determine the limit value of the largest lateral deflection. However, in all studies reported here, many tests and data need to be collected for the methods based on probabilistic calculations. This is a costly business time consuming procedure as well. Therefore, with the help of the convex model which does not require any experiment, the results obtained by obtaining the critical buckling load equations of the columns with uncertainty are advantageous compared to the approaches such as probabilistic calculations.

The convex model is a model used to obtain the equation that gives the minimum and maximum critical buckling loads at different levels of uncertainty in non-deterministic applications. This model has been applied to exact solutions such as Least-Square Vector Machine (LSSVM) [Zhang et al., 2017], nonlocal plate theory [Radebe and Adali, 2014] and cosinusoidal polynomials [Verhaeghe et al., 2013]. Radebe and Adali reported a study based on uncertainty on a nano-sized plate due to the possible defects that can be made since the material properties at the nano level can vary during production and at the same time it is difficult to measure the dimensions. They obtained a certain critical buckling load equation using nonlocal plate theory and then applied a convex model. Lagrange multipliers were used to calculate the uncertain parameters. Thus, a very useful equation is obtained which provides the minimum and maximum critical buckling loads at different levels of uncertainty. They determined the effect of small-scale changes on the critical buckling load at different levels of uncertainty [Radebe and Adali, 2014]. In another study, Oktem and Adali used the convex model to determine the minimum buckling load at different uncertainty levels in the nanocomposite columns where the material properties were defined as uncertain. The effects of various material parameters on buckling behavior at different levels of uncertainty were investigated. In this study, it has been observed that there are significant losses in critical buckling load as a result of slight changes in the parameters [Oktem and Adali, 2018]. Radebe and Adali determined the buckling behavior of a composite cylinder in which the material properties were defined as uncertain under externally applied pressure by using a convex model. For hybrid cross-ply cylinders with high rigidity skin and low rigidity core layers, the most cost-effective design problem was examined with material properties that differed around the nominal values, resulting in a very useful buckling equation for engineers in analytically expressed closed form [Radebe and Adali, 2014]. Bi et al. using the convex model and the LSSVM method, they examined whether the cylindrical shells with laminated and functionally graded properties were robust or not. They concluded that even minor changes in the level of uncertainty led to intense fluctuations in critical buckling load. Considering the studies conducted using convex model, it's seen that it is inevitable that small differences in material properties and dimensions which cause changes in critical buckling load should be included in the calculations of buckling [Bi et al., 2013].

Literature review shows that buckling loads are important in columns with the subject of uncertainty. There is a preliminary study that Kaya and Oktem reported that the material properties and dimensions are considered as uncertain in the isotropic columns in the limited cross-sections of columns and that the critical buckling equations are used analytically by using the convex model [Kaya and Oktem, 2018] and also there is no study in which the critical buckling load is examined with the uncertainty of the material properties and dimensions by using convex model which is investigated in this study for the composite structures.

In this work, analytical buckling equations in closed form are presented for the orthotropic composite I and L sections for different uncertainty levels. Thus, elastic buckling behavior can be estimated in early design stages before the production stage. By using the sensitivity analysis, it can also be revealed which parameters have the most important effects on the lowest critical buckling loads at different uncertainty levels. Knowing all critical buckling values closer to reality under the defined conditions will help to minimize human and material losses. Particularly this allow engineers and scientists to obtain the most appropriate design.

METHOD

For composite columns, instead of using classical lamination theory (CLT), buckling analysis is carried out by taking into account the effects of transverse shear deformations with the aid of first-order shear deformation theory (FSDT), which is especially important in buckling analysis of laminated composites [Berthelot, 1999].

Buckling equations are derived for orthotropic columns and they are given as below:

$$hG_{13}\left(\frac{d\varphi_x}{dx}+\frac{d^2\omega_0}{dx^2}\right)-N_0\frac{d^2\omega_0}{dx^2}=0 \tag{1}$$

$$\left(\frac{D_{11}D_{22}-D_{12}^2}{D_{22}}\right)\frac{d^2\varphi_x}{dx^2}-hG_{13}\left(\varphi_x+\frac{d\omega_0}{dx}\right)=0 \tag{2}$$

where h and N_0 refers to the thickness of the rectangular cross-section and applied load, respectively. Multiplying the expressions (1) and (2) by width of the cross-section (b), equations become as follows:

$$A_{cs}G_{13}\left(\frac{d\varphi_x}{dx}+\frac{d^2\omega_0}{dx^2}\right)-P\frac{d^2\omega_0}{dx^2}=0 \tag{3}$$

$$b\left(\frac{D_{11}D_{22}-D_{12}^2}{D_{22}}\right)\frac{d^2\varphi_x}{dx^2}-A_{cs}G_{13}\left(\varphi_x+\frac{d\omega_0}{dx}\right)=0 \tag{4}$$

The expressions satisfying the boundary conditions for the edges of the column are written as:

$$\varphi_x = A\cos\frac{m\pi x}{L} \tag{5}$$

$$\omega_0 = B\sin\frac{m\pi x}{L} \tag{6}$$

Substituting expressions (7) and (8) in equations (3) and (4), we obtain:

$$A\left(-\frac{m\pi}{L}A_{cs}G_{13}\right)+B\left(\frac{m^2\pi^2}{L^2}\left(P-A_{cs}G_{13}\right)\right)=0 \tag{7}$$

$$A\left(-\frac{m^2\pi^2}{L^2}bD-A_{cs}G_{13}\right)+B\left(-\frac{m\pi}{L}A_{cs}G_{13}\right)=0 \tag{8}$$

A nonzero solution exists in the case where the determinant of the matrix of coefficients *A* and *B* is zero, which leads to:

$$P_{cr}=\frac{m^2\pi^2 A_{cs}\,G_{13}\,b\,D}{m^2\pi^2\,b\,D+A_{cs}\,G_{13}\,L^2} \tag{9}$$

Deterministic critical buckling load equation for composite columns [Berthelot, 1999] is shown in equation (9) where A_{cs}, G_{13}, b, L and m refers to the area of cross-section, in plane (1-3) shear modulus, width of the section, the length of column and the buckling mode shape (m=1 for fundamental mode), respectively. D is one of the most important parameters that differ in each column type and therefore affects the buckling because it contains terms of the moment of inertia (Eq. 10).

$$D = \frac{D_{11} D_{22} - D_{12}^2}{D_{22}} \tag{10}$$

The expressions D_{11}, D_{12} and D_{22} are elements of the bending stiffness matrix linking bending moment components to plate curvature. Classical Lamination Theory, based on Kirchoff's hypothesis, gives close results when compared to experimental data in thin columns. However, when the ratio of the width to the thickness is less than 10, it gives results that are not accurate [Turvey, 1995; Berthelot, 1999; Jones, 1999; Reddy, 1999a, 2003b, 2006c; Altenbach and Becker, 2003]. Therefore, in this study, the first-order shear deformation theory is used for all the cross-sections where the critical buckling calculation is performed by using uncertainty for composite columns.

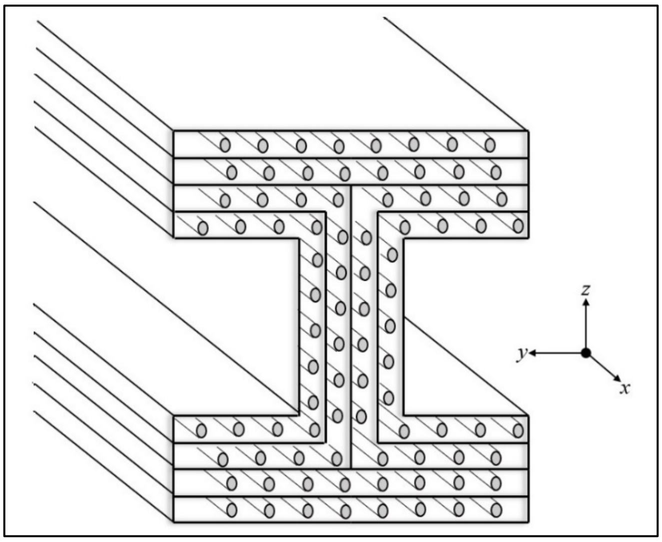

Figure 1: Demonstration of laminated composite column with I-section which has symmetrical stacking sequence $[0°/0°]_s$.

In this study, composite columns columns with I and L cross-sections have the same stacking sequence and the derivations of equations are obtained by using the same stacking sequence.

In uncertainty analysis, the parameters which defined as uncertain by the column types may vary. The modulus of elasticity acting in the axial direction (E_1), the shear modulus (G_{13}) in the plane (1-3), which is expressed in the equations due to the first order shear deformation theory. The parameters that may arise from the inability to obtain the desired thicknesses (h) in the laminates formed by overlapping and column length (L) are also defined as uncertain.

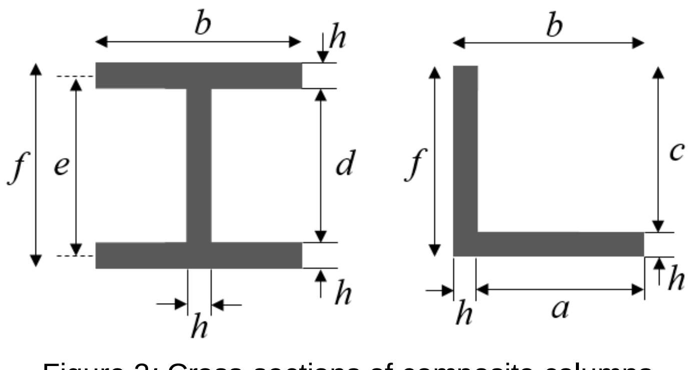

Figure 2: Cross-sections of composite columns

The critical buckling equations with uncertainty analysis are then expressed as follows:

$$\overline{P}_{cr} = \frac{n^2 \pi^2 \overline{E}\,\overline{I}}{\overline{L}^2} \tag{11}$$

$$\overline{P}_{cr} = \frac{\pi^2 \overline{A}_{cs}\, \overline{G}_{13}\, b\, \overline{D}}{\pi^2\, b\, \overline{D} + \overline{A}_{cs}\, \overline{G}_{13}\, \overline{L}^2} \tag{12}$$

Here, over-bar defines the parameters which are taken as uncertain for all the equations in this study. Convex analysis is used to obtain the minimum point of critical buckling load. The critical buckling load equation (P_{cr}), expressed by uncertain parameters, is minimized by the following inequality boundary equation according to the uncertain parameters (δ_i) for minimizing the critical buckling load by considering the worst-case scenario:

$$\sum_{i=1}^{n} \delta_i^{\,2} \le \beta^{\,2} \tag{13}$$

As the lowest critical buckling load at different levels of uncertainty is possible when the δ_i parameters are located on the surface of the ellipsoid, the constraint equation is expressed as follows:

$$\sum_{i=1}^{n} \delta_i^{\,2} - \beta^{\,2} = 0 \tag{14}$$

The method of Lagrange multipliers is a very functional method that gives the minimum and maximum values of a multivariable function subject to restriction and is often used in optimization studies in mechanical fields [Stoecker, 1989; Haftka et al., 1990]. The critical buckling load and the constraint equation are expressed depending on the δ_i variables and the expression that gives the critical buckling load at the point where these points are tangent to the limiting curve is obtained. Thus, the minimum critical buckling load is obtained. In this context, the equation in which the Lagrange method is applied and it is expressed as follows:

$$L(\delta_i, \lambda) = \overline{P}_{cr} + \lambda \left(\sum_{i=1}^{n} \delta_i^2 - \beta^2 \right) \tag{15}$$

Here, β refers to the level of uncertainty. When β is equal to zero, the equation becomes deterministic. Using the Lagrange multipliers method, the equality of all δ values in terms of β is expressed and thus the equation giving the critical buckling load is determined as a function dependent on the level of uncertainty (β). The analytical equation results in the variation of the minimum critical buckling load for different levels of uncertainty (β) are then obtained.

Sensitivity analysis of uncertain parameters is examined using the following expression:

$$S(\delta_i) = \left|\frac{\partial P_{cr}(\beta)}{\partial \delta_i}\right| \frac{|\delta_i|}{P_{cr}(0)} \tag{16}$$

Sensitivity analysis shows which uncertain parameter has the greatest unfavorable effect on critical buckling load. Knowing unfavorable effects of the parameters in the critical buckling provide that which parameter should be paid more attention especially in the preliminary design stages.

The results obtained by the finite element method are compared to the data of the analytical critical buckling equations based on the different level of uncertainty. Critical buckling load values are consistent with each other and examined in detail in the results section. Analysis performed in ANSYS Workbench environment is carried out by defining the composite materials as a shell in ACP (ANSYS Composite PrepPost) module. The 'Eigenvalue Buckling' module is used for all buckling analyzes.

DERIVATIONS OF EQUATIONS

In this section, minimum critical buckling load equations for I and L cross-sections are derived by using convex modeling with the aid of "Lagrange Multipliers Method" as mentioned in previous section.

Laminated Orthotropic Composite I-Section Column

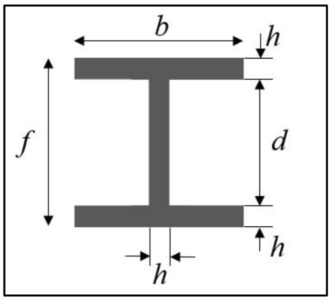

Figure 3*:* Composite I-section column.

For composite I-section column (Figure 3), the uncertain material properties and dimensions are defined as:

$$\overline{E}_1 = E_1(1+\delta_1) \tag{17}$$

$$\overline{G}_{13} = G_{13}(1+\delta_2) \tag{18}$$

$$\overline{L} = L(1+\delta_3) \tag{19}$$

$$\overline{h} = h(1+\delta_4) \tag{20}$$

$$\overline{d} = f - 2h(1+\delta_4) \tag{21}$$

$$\overline{A}_{cs} = h\left(1+\delta_4\right)\left[2b+\left[f-2h\left(1+\delta_4\right)\right]\right] \tag{22}$$

$$y_c = \frac{1}{2}b \tag{23}$$

$$z_c = \frac{1}{2}f \tag{24}$$

$$\overline{\nu}_{21} = \frac{\nu_{12}E_2}{E_1\left(1+\delta_1\right)} \tag{25}$$

Substituting expression (25) into transformed reduced stiffness matrix equations for $[0/0]_s$ stacking sequence, expressions (26) – (28) are obtained as:

$$\overline{Q}_{11,u} = \frac{E_1^2\left(1+\delta_1\right)^2}{-E_2\nu_{12}^2+E_1\delta_1+E_1} \tag{26}$$

$$\overline{Q}_{12,u} = \frac{\nu_{12}E_2E_1\left(1+\delta_1\right)}{-E_2\nu_{12}^2+E_1\delta_1+E_1} \tag{27}$$

$$\overline{Q}_{22,u} = \frac{E_2E_1\left(1+\delta_1\right)}{-E_2\nu_{12}^2+E_1\delta_1+E_1} \tag{28}$$

Equation (29) represents the area of moment of inertia of composite I-section column using parallel-axis theorem.

$$\overline{I}_{yy} \cong \frac{1}{12}h\left(1+\delta_4\right)\begin{bmatrix}\delta_4\left(-24h^3+8h^2\left(2b+3f\right)-6hf\left(2b+f\right)\right)+f^3\\+\left(b-h\right)\left[6f^2-4h\left(3f-2h\right)\right]\end{bmatrix} \tag{29}$$

The bending stiffness matrices are defined as:

$$\overline{D}_{11} = \frac{\overline{I}_{yy}}{b}\overline{Q}_{11,u} \tag{30}$$

$$\overline{D}_{11} \cong \frac{1}{12}\frac{E_1^2 h(1+\delta_4)(1+\delta_1)^2}{b\left(-E_2\nu_{12}^2+E_1\delta_1+E_1\right)}\left[\begin{array}{l}\delta_4\begin{pmatrix}-24h^3+8h^2(2b+3f)\\ -6hf(2b+f)\end{pmatrix}+f^3\\ +(b-h)\left[6f^2-4h(3f-2h)\right]\end{array}\right] \tag{31}$$

$$\overline{D}_{12} = \frac{\overline{I}_{yy}}{b}\overline{Q}_{12,u} \tag{32}$$

$$\overline{D}_{12} = \frac{1}{12}\frac{E_2 E_1 h\nu_{12}(1+\delta_1)(1+\delta_4)}{b\left(-E_2\nu_{12}^2+E_1\delta_1+E_1\right)}\left[\begin{array}{l}\delta_4\begin{pmatrix}-24h^3+8h^2(2b+3f)\\ -6hf(2b+f)\end{pmatrix}+f^3\\ +(b-h)\left[6f^2-4h(3f-2h)\right]\end{array}\right] \tag{33}$$

$$\overline{D}_{22} = \frac{\overline{I}_{yy}}{b}\overline{Q}_{22,u} \tag{34}$$

$$\overline{D}_{22} = \frac{1}{12}\frac{E_2 E_1 h(1+\delta_1)(1+\delta_4)}{b\left(-E_2\nu_{12}^2+E_1\delta_1+E_1\right)}\left[\begin{array}{l}\delta_4\begin{pmatrix}-24h^3+8h^2(2b+3f)\\ -6hf(2b+f)\end{pmatrix}+f^3\\ +(b-h)\left[6f^2-4h(3f-2h)\right]\end{array}\right] \tag{35}$$

$$\overline{D} = \overline{D}_{11} - \frac{\overline{D}_{12}^2}{\overline{D}_{22}} \tag{36}$$

$$\overline{D} = \frac{1}{12}\frac{E_1 h(1+\delta_1)(1+\delta_4)}{b}\left[\begin{array}{l}\delta_4\begin{pmatrix}-24h^3+8h^2(2b+3f)\\ -6hf(2b+f)\end{pmatrix}+f^3\\ +(b-h)\left[6f^2-4h(3f-2h)\right]\end{array}\right] \tag{37}$$

By linearizing the critical buckling load equation based on FSDT with uncertain material properties and dimensions, equation (38) can be obtained as:

$$\overline{P}_{cr,I}^{c} \cong \frac{\varphi_{0,I}^{c}\left(1+\phi_{1,I}^{c}\delta_1+\phi_{2,I}^{c}\delta_2+\phi_{4,I}^{c}\delta_4\right)}{\zeta_{0,I}^{c}(1+\psi_{1,I}^{c}\delta_1+\psi_{2,I}^{c}\delta_2+\psi_{3,I}^{c}\delta_3+\psi_{4,I}^{c}\delta_4)} \tag{38}$$

In order to eliminate the higher order δ_i terms, equation (38) is linearized again and defined as:

$$\overline{P}_{cr,I}^{c} \cong \kappa_I^c\left[1+\left(\phi_{1,I}^c-\psi_{1,I}^c\right)\delta_1+\left(\phi_{2,I}^c-\psi_{2,I}^c\right)\delta_2-\psi_{3,I}^c\delta_3+\left(\phi_{4,I}^c-\psi_{4,I}^c\right)\delta_4\right] \quad (39)$$

where

$$\kappa_I^c=\frac{\varphi_{0,I}^c}{\zeta_{0,I}^c} \quad (40)$$

$$\phi_{1,I}^c=\frac{\varphi_{1,I}^c}{\varphi_{0,I}^c},\ \phi_{2,I}^c=\frac{\varphi_{2,I}^c}{\varphi_{0,I}^c},\ \phi_{4,I}^c=\frac{\varphi_{4,I}^c}{\varphi_{0,I}^c} \quad (41)$$

$$\psi_{1,I}^c=\frac{\zeta_{1,I}^c}{\zeta_{0,I}^c},\ \psi_{2,I}^c=\frac{\zeta_{2,I}^c}{\zeta_{0,I}^c},\ \psi_{3,I}^c=\frac{\zeta_{3,I}^c}{\zeta_{0,I}^c},\ \psi_{4,I}^c=\frac{\zeta_{4,I}^c}{\zeta_{0,I}^c} \quad (42)$$

where φ_i and ζ_i are given in Appendix A. The expression (39) subjected to constraint equation (14) is minimized by using equations (43) - (45).

$$L(\delta_i,\lambda)=\overline{P}_{cr,I}^{c}+\lambda_I^c\left(\sum_{i=1}^{4}\delta_i^2-\beta^2\right) \quad (43)$$

$$\frac{\partial}{\partial\delta_i}L=0\ \ (i=1,2,3,4) \quad (44)$$

$$\frac{\partial}{\partial\lambda_I^c}L=0 \quad (45)$$

$$\lambda_I^c=\pm\frac{1}{2}\frac{\kappa_I^c\omega_I^c}{\beta} \quad (46)$$

where

$$\omega_I^c=\sqrt{A_I^c+B_I^c-2\psi_{1,I}^c\phi_{1,I}^c-2\psi_{2,I}^c\phi_{2,I}^c-2\psi_{4,I}^c\phi_{4,I}^c}$$

$$A_I^c=\left(\psi_{1,I}^c\right)^2+\left(\psi_{2,I}^c\right)^2+\left(\psi_{3,I}^c\right)^2+\left(\psi_{4,I}^c\right)^2 \quad (47)$$

$$B_I^c=\left(\phi_{1,I}^c\right)^2+\left(\phi_{2,I}^c\right)^2+\left(\phi_{4,I}^c\right)^2$$

By substituting the expression (46) which leads to minimize or maximize the critical buckling load into equations (44), expressions of δ_i are obtained as below:

$$\delta_1 = \frac{\left(\psi_{1,I}^{c} - \phi_{1,I}^{c}\right)\beta}{\omega_I^c} \tag{48}$$

$$\delta_2 = \frac{\left(\psi_{2,I}^{c} - \phi_{2,I}^{c}\right)\beta}{\omega_I^c} \tag{49}$$

$$\delta_3 = \frac{\psi_{3,I}^{c}\beta}{\omega_I^c} \tag{50}$$

$$\delta_4 = \frac{\left(\psi_{4,I}^{c} - \phi_{4,I}^{c}\right)\beta}{\omega_I^c} \tag{51}$$

Laminated Orthotropic Composite L-Section Column

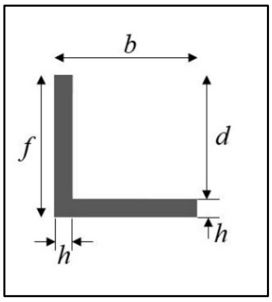

Figure 4: Composite L-section column.

For composite L-section column (Figure 4), the uncertain material properties and dimensions are defined as:

$$\overline{E}_1 = E_1\left(1+\delta_1\right) \tag{52}$$

$$\overline{G}_{13} = G_{13}\left(1+\delta_2\right) \tag{53}$$

$$\overline{L} = L\left(1+\delta_3\right) \tag{54}$$

$$\overline{h} = h\left(1+\delta_4\right) \tag{55}$$

$$\overline{d} = f - h\left(1+\delta_4\right) \tag{56}$$

$$\overline{A}_{CS} = h\left(1+\delta_4\right)\left[b+\left[f-h\left(1+\delta_4\right)\right]\right] \tag{57}$$

$$\overline{y}_c = \frac{1}{2}\frac{-h^2\delta_4^2 + fh\delta_4 - 2h^2\delta_4 + b^2 + fh - h^2}{-h\delta_4 + b + f - h} \tag{58}$$

$$\overline{z}_c = \frac{1}{2}\frac{-h^2\delta_4^2 + bh\delta_4 - 2h^2\delta_4 + bh + f^2 - h^2}{-h\delta_4 + b + f - h} \tag{59}$$

$$\overline{y}_c \cong \frac{1}{2}\frac{\left(-h^2\delta_4^2 + fh\delta_4 - 2h^2\delta_4 + b^2 + fh - h^2\right)}{b + f - h}\left(1+\frac{h}{b+f-h}\delta_4\right) \tag{60}$$

$$\overline{z}_c \cong \frac{1}{2}\frac{\left(-h^2\delta_4^2 + bh\delta_4 - 2h^2\delta_4 + bh + f^2 - h^2\right)}{b + f - h}\left(1+\frac{h}{b+f-h}\delta_4\right) \tag{61}$$

$$\overline{\nu}_{21} = \frac{\nu_{12}E_2}{E_1\left(1+\delta_1\right)} \tag{62}$$

Substituting expression (62) into transformed reduced stiffness matrix equations for $[0/0]_s$ stacking sequence, expressions (63) – (65) are obtained as:

$$\overline{Q}_{11,u} = \frac{E_1^2\left(1+\delta_1\right)^2}{-E_2\nu_{12}^2 + E_1\delta_1 + E_1} \tag{63}$$

$$\overline{Q}_{12,u} = \frac{\nu_{12}E_2E_1\left(1+\delta_1\right)}{-E_2\nu_{12}^2 + E_1\delta_1 + E_1} \tag{64}$$

$$\overline{Q}_{22,u} = \frac{E_2E_1\left(1+\delta_1\right)}{-E_2\nu_{12}^2 + E_1\delta_1 + E_1} \tag{65}$$

Equation (66) represents the area of moment of inertia of composite L-section column using parallel-axis theorem.

$$\overline{I}_{yy} = \left(\frac{b\overline{h}^3}{12} + b\overline{h}\left(\overline{z}_c - \frac{\overline{h}}{2}\right)^2\right) + \left(\frac{\overline{h}\,\overline{d}^3}{12} + \overline{d}\,\overline{h}\left(\frac{\overline{d}}{2} + \overline{h} - \overline{z}_c\right)^2\right) \tag{66}$$

The bending stiffness matrices are defined as:

$$\overline{D}_{11} = \frac{\overline{I}_{yy}}{b}\overline{Q}_{11,u} \tag{67}$$

$$\overline{D}_{12} = \frac{\overline{I}_{yy}}{b}\overline{Q}_{12,u} \tag{68}$$

$$\overline{D}_{22} = \frac{\overline{I}_{yy}}{b}\overline{Q}_{22,u} \tag{69}$$

$$\overline{D} = \overline{D}_{11} - \frac{\overline{D}_{12}^{2}}{\overline{D}_{22}} \tag{70}$$

$$\overline{D} \cong \frac{1}{12}\frac{E_1 h}{b\left(b+f-h\right)^2}\left[\begin{matrix}\left(1+\delta_1\right)\left[\left(b+f-h\right)\left[\begin{matrix}h^4-2h^3\left(b+2f\right)\\+h^2\left(b^2+4bf+6f^2\right)\\-2hf^2\left(3b+2f\right)\\+f^3\left(4b+f\right)\end{matrix}\right]\right]+\\ \delta_4\left[\begin{matrix}-4h^5+h^4\left(11b+17f\right)\\-2h^3\left(5b^2+16bf+14f^2\right)\\+h^2\left(3b^3+15b^2f+36bf^2+22f^3\right)\\-4f^2\left(b+f\right)\left[h\left(3b+2f\right)-f\left(b+\frac{1}{4}f\right)\right]\end{matrix}\right]\end{matrix}\right] \tag{71}$$

By linearizing the critical buckling load equation based on FSDT with uncertain material properties and dimensions, equation (72) can be obtained as:

$$\overline{P}_{cr,L}^{c} \cong \frac{\varphi_{0,L}^{c}\left(1+\phi_{1,L}^{c}\delta_1+\phi_{2,L}^{c}\delta_2+\phi_{4,L}^{c}\delta_4\right)}{\zeta_{0,L}^{c}\left(1+\psi_{1,L}^{c}\delta_1+\psi_{2,L}^{c}\delta_2+\psi_{3,L}^{c}\delta_3+\psi_{4,L}^{c}\delta_4\right)} \tag{72}$$

In order to eliminate the higher order δ_i terms, equation (72) is linearized again and defined as:

$$\overline{P}_{cr,L}^{c} \cong \kappa_L^c\left[1+\left(\phi_{1,L}^{c}-\psi_{1,L}^{c}\right)\delta_1+\left(\phi_{2,L}^{c}-\psi_{2,L}^{c}\right)\delta_2-\psi_{3,L}^{c}\delta_3+\left(\phi_{4,L}^{c}-\psi_{4,L}^{c}\right)\delta_4\right] \tag{73}$$

where

$$\kappa_L^c = \frac{\varphi_{0,L}^c}{\zeta_{0,L}^c} \tag{74}$$

$$\phi_{1,L}^c = \frac{\varphi_{1,L}^c}{\varphi_{0,L}^c}, \ \phi_{2,L}^c = \frac{\varphi_{2,L}^c}{\varphi_{0,L}^c}, \ \phi_{4,L}^c = \frac{\varphi_{4,L}^c}{\varphi_{0,L}^c} \tag{75}$$

$$\psi_{1,L}^c = \frac{\zeta_{1,L}^c}{\zeta_{0,L}^c}, \ \psi_{2,L}^c = \frac{\zeta_{2,L}^c}{\zeta_{0,L}^c}, \ \psi_{3,L}^c = \frac{\zeta_{3,L}^c}{\zeta_{0,L}^c}, \ \psi_{4,L}^c = \frac{\zeta_{4,L}^c}{\zeta_{0,L}^c} \tag{76}$$

where φ_i and ζ_i are given in Appendix A. The expression (73) subjected to constraint equation (14) is minimized by using equations (77) - (79).

$$L(\delta_i, \lambda) = \overline{P}_{cr,L}^c + \lambda_L^c \left(\sum_{i=1}^{4} \delta_i^2 - \beta^2 \right) \tag{77}$$

$$\frac{\partial}{\partial \delta_i} L = 0 \ \ (i = 1, 2, 3, 4) \tag{78}$$

$$\frac{\partial}{\partial \lambda} L = 0 \tag{79}$$

$$\lambda_L^c = \pm \frac{1}{2} \frac{\kappa_L^c \omega_L^c}{\beta} \tag{80}$$

where

$$\omega_L^c = \sqrt{A_L^c + B_L^c - 2\psi_{1,L}^c \phi_{1,L}^c - 2\psi_{2,L}^c \phi_{2,L}^c - 2\psi_{4,L}^c \phi_{4,L}^c}$$

$$A_L^c = \left(\psi_{1,L}^c\right)^2 + \left(\psi_{2,L}^c\right)^2 + \left(\psi_{3,L}^c\right)^2 + \left(\psi_{4,L}^c\right)^2 \tag{81}$$

$$B_L^c = \left(\phi_{1,L}^c\right)^2 + \left(\phi_{2,L}^c\right)^2 + \left(\phi_{4,L}^c\right)^2$$

By substituting the expression (80) which leads to minimize or maximize the critical buckling load into equations (78), expressions of δ_i are obtained as below:

$$\delta_1 = \frac{\left(\psi_{1,L}^{c} - \phi_{1,L}^{c}\right)\beta}{\omega_L^{c}} \tag{82}$$

$$\delta_2 = \frac{\left(\psi_{2,L}^{c} - \phi_{2,L}^{c}\right)\beta}{\omega_L^{c}} \tag{83}$$

$$\delta_3 = \frac{\psi_{3,L}^{c}\beta}{\omega_L^{c}} \tag{84}$$

$$\delta_4 = \frac{\left(\psi_{4,L}^{c} - \phi_{4,L}^{c}\right)\beta}{\omega_L^{c}} \tag{85}$$

RESULTS AND DISCUSSIONS

Table 1 represents the values of material properties and the dimensions used in this study for the composite columns. The mechanical properties of carbon/epoxy are used in order to obtain the uncertain critical buckling load values at defined level of uncertainty.

Table 1*:* Nominal dimensions and material properties for composite columns.

Column Type	**Material Properties (GPa)**		**Dimensions (mm)**			
	E_1	*G_{13}*	*b*	*f*	*h*	*L*
I	*121*	*4.7*	*30*	*30*	*4*	*1000*
L	*121*	*4.7*	*30*	*30*	*4*	*1000*

In the graph (Figure 5), the sections with the losses in normalized critical buckling load according to different levels of uncertainty of composite columns are represented. It’s seen that even minor fluctuations on the uncertainty level result in considerable loss of normalized critical buckling load values.

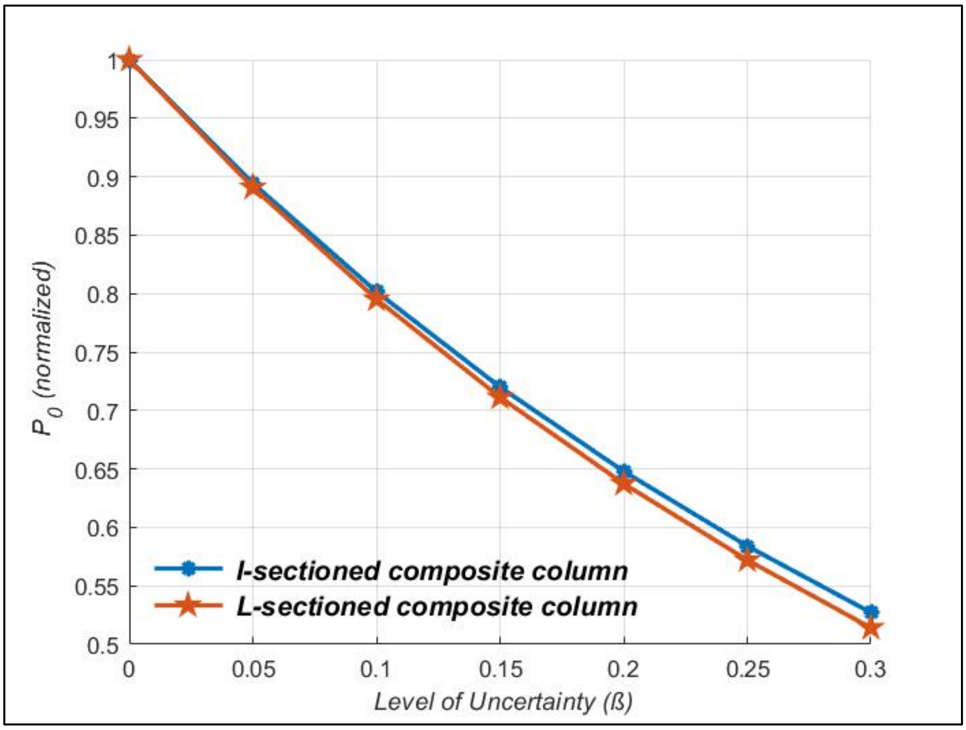

Figure 5: Normalized critical buckling load variations for composite columns.

Figure 6 and Figure 7 represent which parameter has the most unfavorable effect on the critical buckling load. For all composite columns, the parameter G_{13} which refers to shear modulus in 1-3 plane has almost no effect on critical buckling load with the increasing level of uncertainty for defined uncertain parameters.

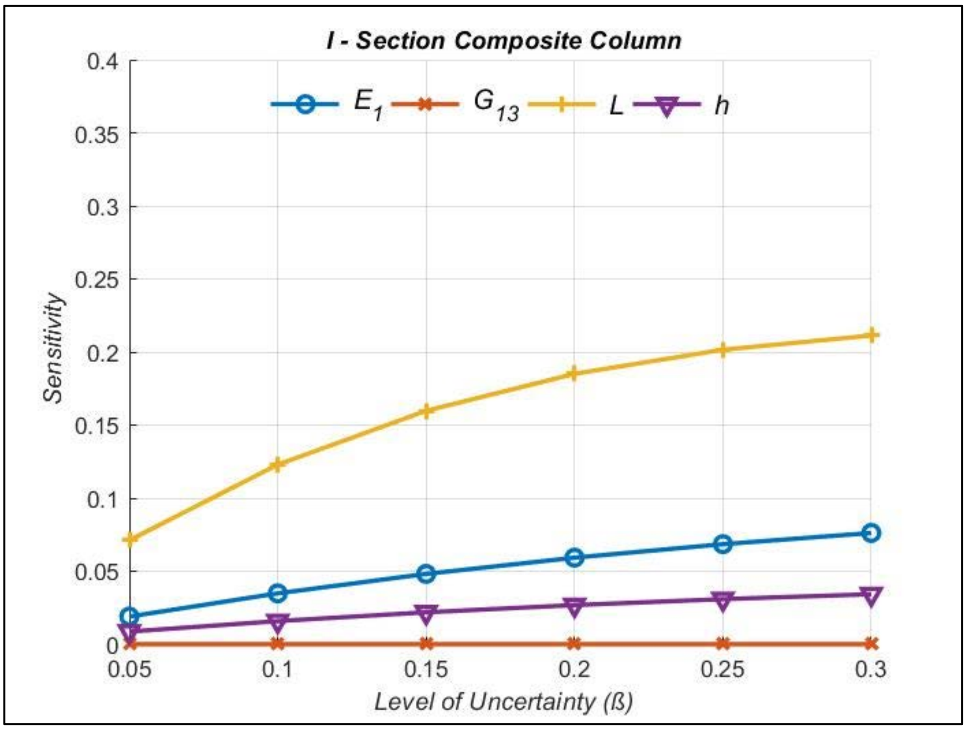

Figure 6: Sensitivity analysis of I-section composite column.

The graphs (Figure 6 - Figure 7) also indicate that the column length (*L*), by far, is the most negatively affected parameter regarding the critical buckling load. The results clearly show that the E_1 and thickness of cross-section (*h*) also play a significant role considering the variability on the longitudinal Young's modulus and thickness during manufacturing stages.

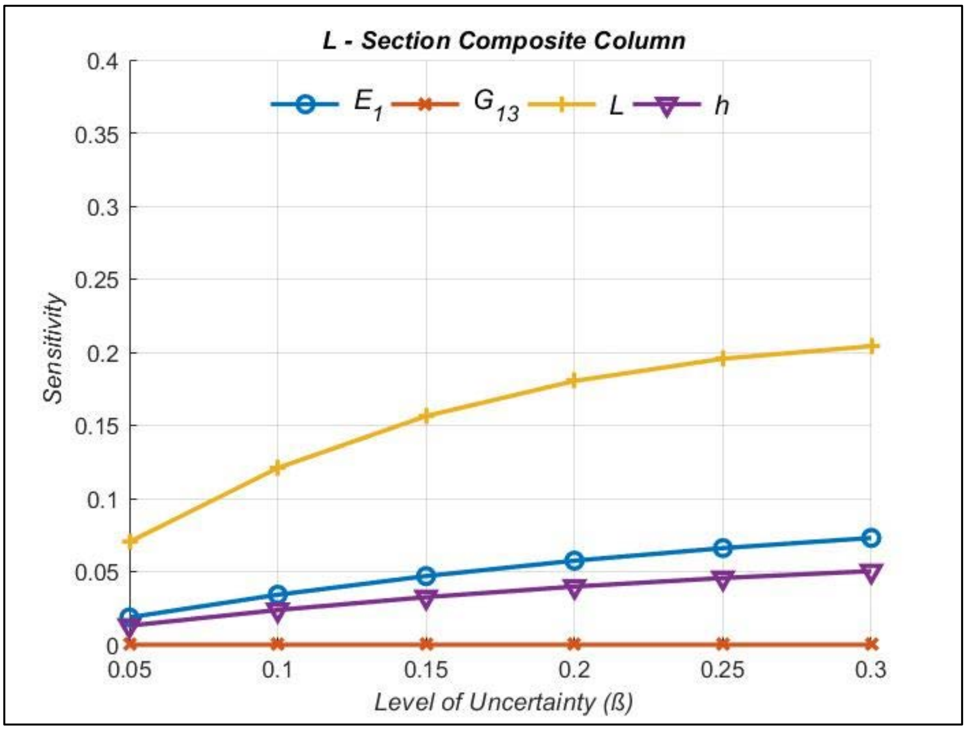

Figure 7: Sensitivity analysis of L-section composite column.

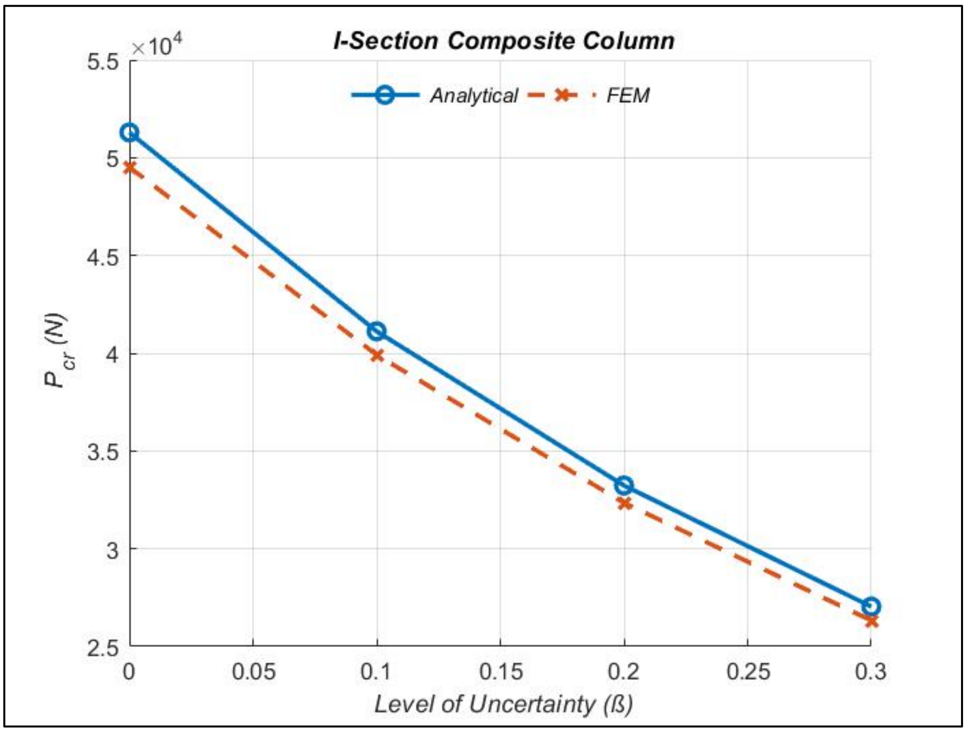

Figure 8: Comparison of analytical and FEM results for I-section composite column.

Figure 8 shows the comparison of the results obtained with analytical and finite element method (FEM). Results provide closer values which validate the derived equations showed in previous chapter.

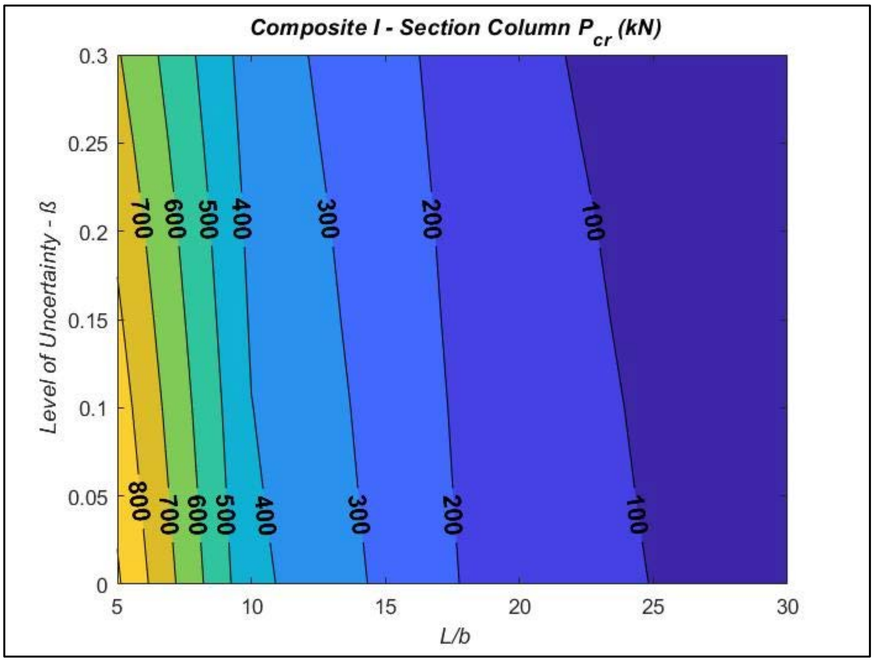

Figure 9: Contour plots of composite I-section column.

Figure 9 and Figure 10 show contour plot of the variation of the critical buckling load at different level of uncertainty with the ratio of *L/b*. Although the increment of L/b values are low for the range between 5 and 10, the decrement values on critical buckling loads are remarkable for all composite sections as it's represented in contour plots (Figure 9 and Figure 10).

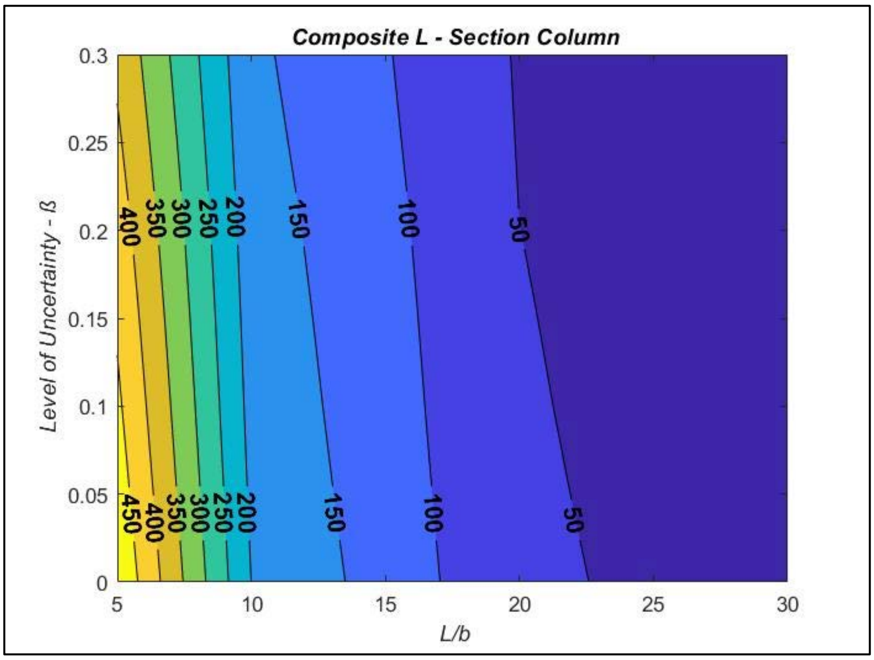

Figure 10: Contour plots of composite L-section column.

CONCLUSIONS

In this study, changes in critical buckling load of composite columns with uncertain material properties and dimensions are obtained. It can be concluded that even cases for the values smaller than 10 of L/b, especially the dimensional uncertainties should be taken into account during production stage although the deviations of uncertain parameters are small. Also, it's clear that the uncertainty unfavorably effects the critical buckling load for all L/b values. Therefore, instead of calculating critical buckling loads using certain values of material properties and its dimensions, the buckling calculations should be made considering the uncertainty which provides to obtain realistic critical buckling loads at defined level of uncertainty. As a result, critical buckling load equations are obtained analytically for I and L cross sections of the composite columns with uncertain material properties and dimensions. It is stated that even small changes in the uncertainty level create significant deviations in critical buckling load. It is also important to state that performing sensitivity analysis is also very important to calculate which parameter has more effect on the losses in critical buckling load. It's observed that the issue of uncertainty has to be included in the calculations particularly at the preliminary design stages.

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Appendix A

$$\varphi_{0,I}^{c} = \frac{1}{12}\pi^2 E_1 G_{13} h^2 \left(-2h+2b+f\right)\begin{bmatrix} 2b\left(3f^2-6fh+4h^2\right) \\ +f\left(f^2-6fh+12h^2\right)-8h^3 \end{bmatrix} \tag{A.1}$$

$$\varphi_{1,I}^{c} = \frac{1}{12}\pi^2 E_1 G_{13} h^2 \left(-2h+2b+f\right)\begin{bmatrix} 2b\left(3f^2-6fh+4h^2\right) \\ +f\left(f^2-6fh+12h^2\right)-8h^3 \end{bmatrix} \tag{A.2}$$

$$\varphi_{2,I}^{c} = \frac{1}{12}\pi^2 E_1 G_{13} h^2 \left(-2h+2b+f\right)\begin{bmatrix} 2b\left(3f^2-6fh+4h^2\right) \\ +f\left(f^2-6fh+12h^2\right)-8h^3 \end{bmatrix} \tag{A.3}$$

$$\varphi_{4,I}^{c} = 2G_{13}\pi^2 E_1 h^2 \begin{bmatrix} 4h^4-\frac{20}{3}h^3\left(b+f\right)+\frac{8}{3}h^2\left(b+\frac{1}{2}f\right)\left(b+3f\right)+ \\ hf\left(-3b^2-\frac{9}{2}bf-f^2\right)+f^2\left(b+\frac{1}{2}f\right)\left(b+\frac{1}{6}f\right) \end{bmatrix} \tag{A.4}$$

$$\zeta_{0,I}^{c} = 2h\begin{bmatrix} \frac{1}{4}\pi^2 E_1 \begin{bmatrix} -\frac{4}{3}h^3+h^2\left(\frac{4}{3}b+2f\right)+f^2\left(b+\frac{1}{6}f\right) \\ +hf\left(-2b-f\right) \end{bmatrix} \\ +L^2 G_{13}\left(b+\frac{1}{2}f-h\right) \end{bmatrix} \tag{A.5}$$

$$\zeta_{1,I}^{c} = \frac{1}{12}\pi^2 E_1 h\left[2b\left(3f^2-6fh+4h^2\right)+f\left(f^2-6fh+12h^2\right)-8h^3\right] \tag{A.6}$$

$$\zeta_{2,I}^{c} = G_{13}L^2 h\left(2b+f-2h\right) \tag{A.7}$$

$$\zeta_{3,I}^{c} = 2G_{13}L^{2}h\left(2b+f-2h\right) \tag{A.8}$$

$$\zeta_{4,I}^{c} = 2h\left[\frac{1}{4}\pi^{2}E_{1}\left(f-2h\right)^{2}\left(b+\frac{1}{6}f-\frac{4}{3}h\right)+L^{2}G_{13}\left(b+\frac{1}{2}f-2h\right)\right] \tag{A.9}$$

$$\varphi_{0,L}^{c} = \frac{1}{12}\pi^{2}E_{1}G_{13}h^{2}\left[\begin{pmatrix} h^{4}-2h^{3}\left(b+2f\right)+h^{2}\left(b^{2}+4bf+6f^{2}\right) \\ -2hf^{2}\left(3b+2f\right)+f^{3}\left(4b+f\right) \end{pmatrix}\right] \tag{A.10}$$

$$\varphi_{1,L}^{c} = \frac{1}{12}\pi^{2}E_{1}G_{13}h^{2}\left[\begin{pmatrix} h^{4}-2h^{3}\left(b+2f\right)+h^{2}\left(b^{2}+4bf+6f^{2}\right) \\ -2hf^{2}\left(3b+2f\right)+f^{3}\left(4b+f\right) \end{pmatrix}\right] \tag{A.11}$$

$$\varphi_{2,L}^{c} = \frac{1}{12}\pi^{2}E_{1}G_{13}h^{2}\left[\begin{pmatrix} h^{4}-2h^{3}\left(b+2f\right)+h^{2}\left(b^{2}+4bf+6f^{2}\right) \\ -2hf^{2}\left(3b+2f\right)+f^{3}\left(4b+f\right) \end{pmatrix}\right] \tag{A.12}$$

$$\varphi_{4,L}^{c} = \frac{1}{6}\pi^{2}E_{1}G_{13}h^{2}\left[\begin{matrix} 3h^{4}-5h^{3}\left(b+2f\right)+2h^{2}\left(b^{2}+4bf+6f^{2}\right) \\ -3f^{2}\left(3b+2f\right)h+f^{3}\left(4b+f\right) \end{matrix}\right] \tag{A.13}$$

$$\zeta_{0,L}^{c} = \frac{1}{12}\frac{h}{b+f-h}\left[\begin{matrix} \pi^{2}E_{1}\left[\begin{matrix} h^{4}-2h^{3}\left(b+2f\right)+h^{2}\left(b^{2}+4bf+6f^{2}\right) \\ -2hf^{2}\left(+3b+2f\right)+f^{3}\left(4b+f\right) \end{matrix}\right] \\ +12G_{13}L^{2}\left[\left(b+f-h\right)^{2}\right] \end{matrix}\right] \tag{A.14}$$

$$\zeta_{1,L}^{c} = \frac{1}{12}\frac{\pi^{2}E_{1}h}{b+f-h}\left[\begin{matrix} h^{4}-2h^{3}\left(b+2f\right)+h^{2}\left(b^{2}+4bf+6f^{2}\right) \\ -2hf^{2}\left(+3b+2f\right)+f^{3}\left(4b+f\right) \end{matrix}\right] \tag{A.15}$$

$$\zeta_{2,L}^{c} = G_{13}hL^{2}\left(b+f-h\right) \tag{A.16}$$

$$\zeta_{3,L}^{c} = 2G_{13}hL^{2}\left(b+f-h\right) \tag{A.17}$$

$$\zeta_{4,L}^{c} = \frac{1}{12}\frac{h}{k^2}\left[\begin{array}{l} \pi^2 E_1 \left[\begin{array}{l} -4h^5 + h^4\left(11b + 17f\right) - \\ 2h^3\left(5b^2 + 16bf + 14f^2\right) + \\ h^2\left(3b^3 + 15b^2 f + 36bf^2 + 22f^3\right) - \\ 4f^2\left(b+f\right)\left[3h\left(b + \frac{2}{3}f\right) + f\left(b + \frac{1}{4}f\right)\right] \end{array}\right] \\ +12G_{13}L^2\left[\left(b+f-2h\right)\left(b+f-h\right)^2\right] \end{array}\right] \tag{A.18}$$