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FINITE ELEMENT METHOD FOR VIBRATION ANALYSIS OF FUNCTIONALLY GRADED TIMOSHENKO BEAMS

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ABSTRACT

In industry and lots of engineering applications, rotating components, turbines, helicopter blades, rotors belong to large usage area. Design, material properties and dynamic properties of these structures or components are so significant with respect to efficiency. Frequencies and mode shapes are used to identify the dynamic properties of structures. In this study, a theoretical investigation in free vibration of a functionally graded beam (FGB) is presented with using Finite Element Model. It is assumed that material properties vary along the beam thickness according to power law distributions. Timoshenko beam theory is studied and the FGB are modeled according to this theorem. Free vibration analysis of flap wise bending is studied at symmetrical functionally graded beam. The governing equations of motion and boundary conditions are derived on the basis of Hamilton principle. Analytical solutions of the natural frequencies are obtained with finite element method which the properties of FGB distribution shape functions are used for exponential FG beams with clamped-free end supports. MATLAB code is developed to analyze the free vibration of the functionally graded rotating Timoshenko beam. In the process, finite element formulation (FE) is used and the calculated results are validated with the ones in open literature.

INTRODUCTION

Functionally graded materials (FGMs) that are new materials are used to increasing functional performance which will have desired property gradient of the material properties which will have desired property gradient to improve design of important structures such as blades, turbines, rotor.

Design, material properties and dynamic properties of these structures and their components have significant effects on system efficiency. Frequencies and mode shapes are used to determine dynamic properties of these structures.

The aim of this study is to develop a computer code by using finite element method for vibration analyses of a functionally graded helicopter rotor blade whose material properties change in the thickness direction. Functionally graded materials (FGMs) that have gained widespread application are used to increase functional performance which will have desired gradient of the

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material properties. This variation provides continuous stress distribution in the FG structures, whereas discontinuous stress distribution appears in another type of advanced materials, i.e. laminated composites. Material properties of the beam such as elastic modulus, shear modulus, Poisson's ratio, material density which are assumed to change continuously through the thickness direction of the beam, as a function of volume fraction and properties of the materials according to a simple power law.

The concept of Functionally Graded Materials (FGMs) was originated from a group of material scientists in Japan as means of preparing thermal barrier materials (Loy et al., 1999). Functionally graded material (FGM) technology has begun to take place in engineering applications at a near future. Functionally graded materials (FGM) resulting from the development of composite material technology is a new generation material that enables the advanced engineering applications by reflecting the mechanical, physical and chemical properties of the materials it contains. Nowadays, the material properties and material composition of functionally graded materials (FGM) whose production areas are increasing and application fields are increasing day by day with the development of additive manufacturing technology and powder metallurgy are changing throughout the structure. The material change takes place in the form of a gradient connected to a function. Due to these properties, the functional graded materials are used and applied in many different sectors such as aerospace, automotive and medical.

Free vibration properties of a functionally graded beam by using finite element method are studied by Alshorbagy [Alshorbagy, 2007]. Birman research to damage of free and forced vibration on functionally graded cantilever beam [Birman and Byrd, 2011]. Chakraborty developed a new beam finite element for analysis of functionally graded beams [Chakraborty, 2003].

Pure metal or a pure ceramic beams and functionally graded material beams are compared with respect to behavioral difference of static, free vibration and wave propagation problems. Also, exact stiffness matrices were developed earlier for Herman–Mindlin rod [Gopalakrishnan, 2000]. In this approach, the shape functions are not only a function of length of the beam but also depend upon cross-sectional and material properties.

Li presented a unified approach for analyzing the static and dynamic behaviors of functionally graded Timoshenko and Euler-Bernoulli beams [Li, 2008]. Sina studied an analytical method for the free vibration analysis of functionally graded beams [Sina, 2009]. They derived the governing equations of motion using Hamilton's principle and investigated the effects of boundary conditions, volume fraction and shear deformation on the natural frequencies and mode shapes.

In this study, functionally graded blades whose material distribution chanhes in the thickness direction are modeled applying the Euler - Bernoulli and Timoshenko beam theories and vibration analyses are performed. In these studies, beam models with different boundary conditions and material properties are investigated. For developing the mathematical models and for the solution, finite element method (FEM) is used. The blade model formulations are derived for both Euler - Bernoulli and Timoshenko beam theories to inspect the effect of different parameters on the vibration characteristics of the beam. For each beam theory, energy expressions are derived by introducing several explanotary figures and tables. The solution method of the developed code is the finite element method (FEM). After the related displacement fields and shape functions are obtained, element stiffness and mass matrices are derived by considering these energy expressions. In the solution part, effects of several parameters, i.e. rotational speed, material properties, power law index parameter, different boundary conditions, rotary inertia, shear deformation and slenderness ratio are investigated. When the results are compared, it is noticed that the difference between the values and the error rates for each different state and beam theory is acceptable.

METHOD

Beam Model

In this study, functionally graded beam of constant thickness of h with cross-sectional dimensions L and b is shown in Fig.1. Free vibration analysis of a rotating functionally graded beam with clamped-free boundary condition based on the Timoshenko beam theory. Energy expressions are derived for a rotating beam model, which contains two different material properties in different compositions and whose material properties change through the blade thickness.

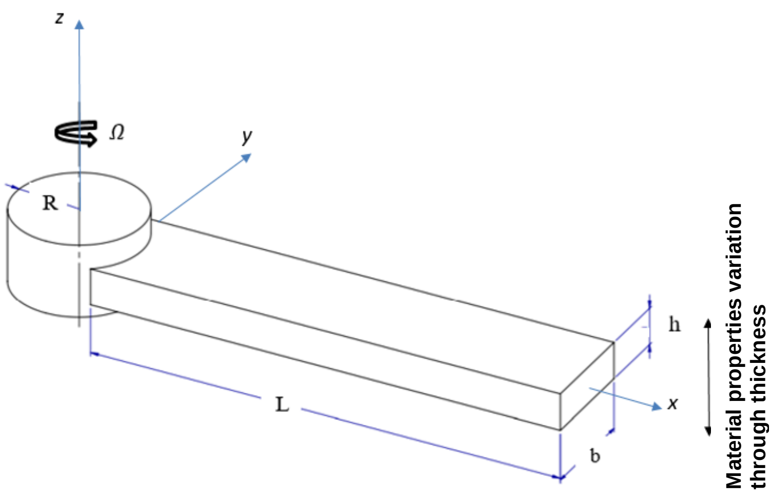

Figure 1. Rotating FG Timoshenko Beam

Functionally Graded Beam Material Properties

The xyz axes represent a global orthogonal coordinate system with its origin at the root of the beam. The beam is assumed to be rotating in the counter-clockwise direction at a constant angular velocity, Ω. In the right-handed Cartesian coordinate system, the x -axis coincides with the neutral axis of the beam in the undeflected position, the z-axis is parallel to the axis of rotation, but not coincident and the y-axis lies in the plane of rotation.

Material properties of the beam, i.e. modulus of elasticity E, shear modulus G, Poisson's ratio, υ and material density, ρ are assumed to vary continuously in the thickness direction z as a function of the volume fraction, and the properties of the constituent materials according to a simple power law. According to the rule of mixture, the effective material property $P(z)$ can be expressed as follows

$$P(z) = P_t V_t + P_b V_b \quad (1)$$

where P_t and P_b are the material properties at the top and bottom surfaces of the beam while V_t and V_b are the corresponding volume fractions. The relation between the volume fractions is given by

$$V_t + V_b = 1 \quad (2)$$

The volume fraction of the top constituent of the beam, V_t, is assumed to be given by

$$V_t = \left(\frac{z}{h} + \frac{1}{2}\right)^n, \quad (n \geq 0) \tag{3}$$

Here n is the non-negative power law index parameter that dictates the material variation profile through the beam thickness.

Considering Eqns. (1)-(3), the effective material properties can be rewritten as follows

$$P(z) = (P_t - P_b)\left(\frac{z}{h} + \frac{1}{2}\right)^n + P_b \tag{4}$$

where P_t and P_b are the material properties at the top and bottom of the beam, respectively and n, is the material nonhomogenity parameter.

It is evident from Eqn.(4) that when $z = h/2$, $E = E_t$, $\nu = \nu_t$, $G = G_t$, $\rho = \rho_t$ and when $z = -h/2$, $E = E_b$, $\nu = \nu_b$, $G = G_b$ and $\rho = \rho_b$.

Energy Expressions for Timoshenko Beam Model

In this section, derivation of the potential and kinetic energy expressions of a rotating Timoshenko Beam are carried out in great detail by using several explanatory figures and tables.
The cross-sectional and the longitudinal views of a Timoshenko beam that undergoes elongation and flapwise bending deflection are given in Fig. 2(a) and Fig. 2(b), respectively. Here, the reference point is chosen and is represented by P_0 before deformation and by P after deformation.

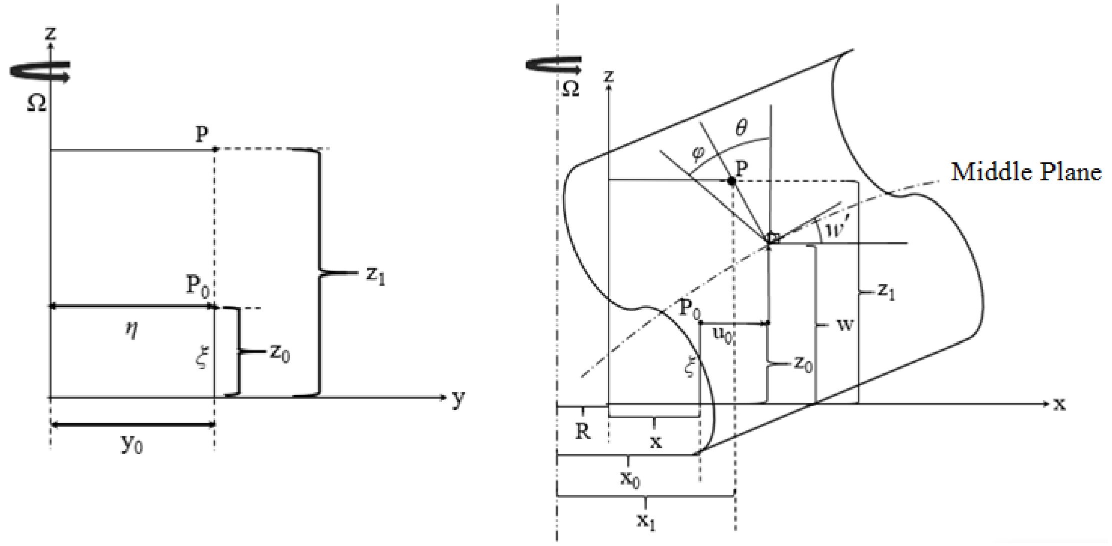

Figure 2. (a) Cross-sectional View (b) Longitudinal View of the Timoshenko Beam

Coordinates of the reference point, P_0 before deformation are given as follows.

$$x_0 = R + x \qquad y_0 = \eta \qquad z_0 = \xi \tag{5}$$

Coordinates of the reference point, P_1 after deformation are given as follows.

$$x_1 = R + x + u_0 - \xi\theta \qquad y_1 = \eta \qquad z_1 = w + \xi \tag{6}$$

The position vectors of the reference point are represented by $\mathbf{r}_0$ and $\mathbf{r}_1$ before and after deflection, respectively. Therefore, $d\mathbf{r}_0$ and $d\mathbf{r}_1$ can be written as follows

$$d\vec{r}_0 = dx\vec{i} + dy\vec{j} + dz\vec{k} \qquad d\vec{r}_1 = (1 + u_0' - \xi\theta')\vec{i} + d\eta\vec{j} + (w'dx + d\xi)\vec{k} \tag{7}$$

where $(\cdot)'$ denotes differentiation with respect to the spanwise coordinate x.

Strain field has to be defined for the reference point to be able to derive the potential and kinetic energy expressions. Therefore, the strain tensor $[\varepsilon ij]$ is obtained as follows (Eringen, 1980).

$$d\vec{r}_1.d\vec{r}_1 - d\vec{r}_0.d\vec{r}_0 = 2\begin{bmatrix} dx & d\eta & d\xi \end{bmatrix}\begin{bmatrix}\varepsilon_{ij}\end{bmatrix}\begin{Bmatrix} dx \\ d\eta \\ d\xi \end{Bmatrix} \tag{8}$$

Here, the strain tensor is

$$\begin{bmatrix}\varepsilon_{ij}\end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{x\eta} & \varepsilon_{x\xi} \\ \varepsilon_{\eta x} & \varepsilon_{\eta\eta} & \varepsilon_{\eta\xi} \\ \varepsilon_{\xi x} & \varepsilon_{\xi\eta} & \varepsilon_{\xi\xi} \end{bmatrix} \tag{9}$$

Substituting Eqn. (7) into Eqn.(8), the components of the strain tensor are obtained as follows

$$\varepsilon_{xx} = u_0' - \xi w'' + \frac{(u_0')^2}{2} + \frac{\xi^2(\theta')^2}{2} - u_0{}'\theta'\xi + \frac{w'^2}{2}, \quad \varepsilon_{x\eta} = 0, \quad \varepsilon_{x\xi} = (w' - \theta) + \xi\theta\theta' - u_0{}'\theta \tag{10}$$

In this work, only ε_{xx}, $\gamma_{x\xi}$ and $\gamma_{x\eta}$ are used in the calculations because for long slender beams, the axial strain ε_{xx} is dominant over the transverse normal strains $\varepsilon_{\eta\eta}$ and $\varepsilon_{\xi\xi}$ Moreover, the shear strain $\gamma_{\xi\eta}$ is by two orders smaller than the other shear strains $\gamma_{x\xi}$ and $\gamma_{x\eta}$. Therefore, $\varepsilon_{\eta\eta}$, $\varepsilon_{\xi\xi}$ and $\gamma_{\xi\eta}$ are neglected (Hodges and Dowell, 1974).

In order to obtain simpler expressions for the strain components given by Eqn. (10), higher order terms can be neglected, so an order of magnitude analysis is performed by using the ordering scheme given by (Ozgumus Ozdemir and Kaya, 2008) and introduced in Table 1.

Table 1. Ordering Scheme for A Timoshenko Beam

$\frac{x}{L} = O(1)$	$\frac{w}{L} = O(\varepsilon)$	$\frac{\eta}{L} = O(\varepsilon)$
$\frac{\xi}{L} = O(\varepsilon)$	$\theta = O(\varepsilon)$	$\frac{u_0}{L} = O(\varepsilon^2)$
$\varphi = w' - \theta = O(\varepsilon^2)$	$u_0' \approx O(\varepsilon^2)$	$\theta' \approx O(\varepsilon^2)$

Applying the ordering scheme to Eqn. (10), the reduced equations are obtained as follows

$$\varepsilon_{xx} = u_0' - \xi w'' + \frac{(u_0')^2}{2} \qquad \varepsilon_{x\eta} = 0 \qquad \varepsilon_{x\xi} = w' - \theta \tag{11}$$

Derivation of The Potential Energy Expression

The potential energy expression is given by

$$U = \frac{1}{2}\int_0^L \left(\iint_A (\sigma_{xx}\varepsilon_{xx} + \tau_{x\xi}\varepsilon_{x\xi}) d\eta d\xi \right) dx \tag{12}$$

The axial force, N and the bending moment. M that act on a laminate at the midplane, Q that act on a laminate at the midplane are expressed as follows (Kollar and Springer, 2003)

$$N = b\int_{-h/2}^{h/2} \sigma dz, \qquad M = b\int_{-h/2}^{h/2} z\sigma dz, \qquad Q = b\int_{-h/2}^{h/2} \tau dz \tag{13}$$

Substituting Eqns. (11) into Eqn. (12) and considering Eqn. (13), the following expression is obtained for the potential energy

$$U = \frac{1}{2}\int_0^L \left[N_x \left\{ u_0' + \frac{(w')^2}{2} \right\} + M_x\theta' + Q_{xz}(w' - \theta) \right] dx \tag{14}$$

where

$$N_x = \bar{A}_{11}u_0' + \bar{B}_{11}\varphi', \qquad M_x = \bar{B}_{11}u_0' + \bar{D}_{11}\varphi' \qquad Q = \bar{A}_{55}\varepsilon_{x\xi} \tag{15}$$

Here the stiffness coefficients are obtained as follows

$$[\bar{A}_{11} \quad \bar{B}_{11} \quad \bar{D}_{11}] = \int_A E(z)\,[1 \quad z \quad z^2]\,dA \qquad \bar{A}_{55} = k\int_A G(z)dA \tag{16}$$

Here, k is the shear correction factor and G is the shear modulus.

Substituting Eqns. (15) into Eqn. (14) and considering Eqn.(16) give

$$U = \frac{1}{2}\int_0^L \left\{ \bar{A}_{11}(u_0')^2 + 2\bar{B}_{11}u_0'\theta' + \bar{D}_{11}(\theta')^2 + \bar{A}_{55}(w' - \theta) \right\} dx \tag{17}$$

The uniform strain ε_0 and the associated axial displacement u_0 due to the centrifugal force $F_{CF}(x)$ is

$$u_0'(x) = \varepsilon_0(x) = \frac{F_{MK}(x)}{EA} = \frac{F_{MK}(x)}{\bar{A}_{11}} \tag{18}$$

where EA is the axial stiffness.

The expression for the centrifugal force is

$$F_{CF}(x) = \int_x^L \rho A\Omega^2(R + x)dx \tag{19}$$

Substituting Eqn. (19) into Eqn. (17) gives the final form of the potential energy as follows

$$U = \frac{1}{2}\int_0^L \left\{ \bar{D}_{11}(\theta')^2 + \bar{A}_{55}(w' - \theta) + F_{MK}(x)(w')^2 \right\} dx \tag{20}$$

Derivation of The Kinetic Energy Expression

The general expression for the kinetic energy is given by

$$T = \frac{1}{2}\int_0^L \int_A \rho(z)\left(V_x^2 + V_y^2 + V_z^2\right) dA\, dx = \frac{b}{2}\int_0^L \int_{-h/2}^{h/2} \rho(z)\left(V_x^2 + V_y^2 + V_z^2\right) d\xi\, dx \quad (21)$$

where $\rho(z)$ is the effective material density.

For a rotating system, the velocity field is defined as follows (Ozgumus Ozdemir and Kaya, 2008).

$$\vec{V} = \frac{\partial \vec{r_1}}{\partial t} + \Omega \vec{k}\, x\, \vec{r_1} \quad (22)$$

The location vector $\vec{r_1}$, i.e. Eqn.(6), is substituted into Eqn.(22) and the velocity components are obtained as follows (Ozgumus Ozdemir and Kaya, 2008)

$$V_x = \dot{u}_0 - \xi\dot{\theta} - \Omega\eta\,, \qquad V_y = (R + x + u_0 - \xi\theta)\Omega\,, \qquad V_z = \dot{w} \quad (23)$$

Substituting the velocity components into Eqn.(21) gives

$$T = \frac{1}{2}\int_0^L (I_1(\dot{u}_0)^2 + I_1(\dot{w})^2 + 2I_2\dot{u}_0\dot{\theta} + I_3(\dot{\theta})^2)dx \quad (24)$$

where I_1, I_2 and I_3 are the inertial characteristics of the beam given as follows

$$[I_1 \quad I_2 \quad I_3] = \int_A \rho(z)\, [1 \quad z \quad z^2]\; dA \quad (25)$$

Additionally, the centrifugal force equation, i.e. Eqn.(19) can be rewritten after considering the definition given in Eqn.(25)

$$F_{CF} = \int_0^L \rho A\Omega^2 \left(R + x\right) dx = \int_0^L I_1\Omega^2\left(R + x\right) dx \quad (26)$$

Finite Element Formulation of Rotating FG Timoshenko Beam

Finite element formulation of a rotating, functionally graded beam that undergoes elongation and flapwise bending deflection is carried out in this section.

The global finite element model of the beam used for the formulation is illustrated in Fig. 3.

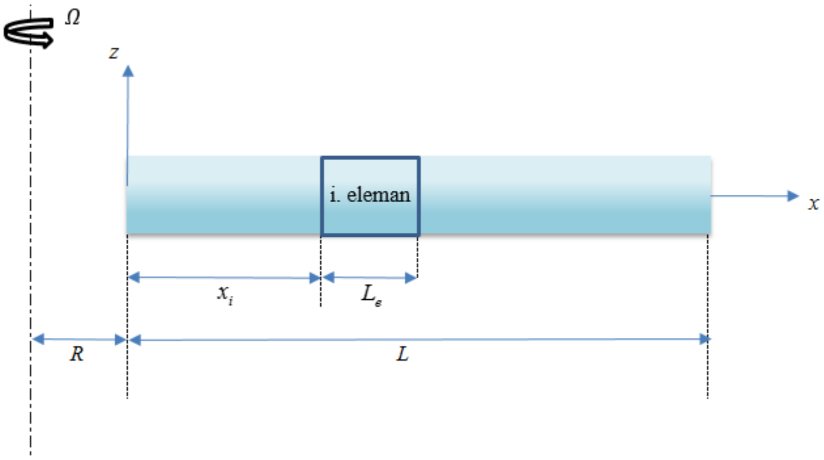

Figure 3: Finite Element Model of A Rotating Functionally Graded Beam

In the case of a rotating beam, additional terms appear in the element matrices due to the centrifugal force. These terms are considered by using finite element formulation for the centrifugal force. Thus, finite element representation of a rotating beam that is given in Fig.4 can be used.

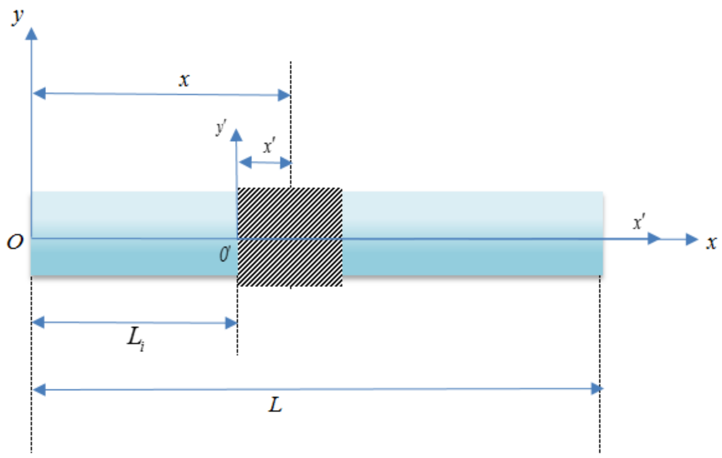

Figure 4: Finite Element Representation of a Rotating Beam

Here L_i is the offset of each element from the rotational axis, XYZ is the global coordinate system while $x'y'z'$ is the local coordinate system.
Referring Fig.4, the centrifugal force given by Eqn.(26) can be expressed in finite element form as follows

$$F_{CF}(x) = I_1\Omega^2\left[R(L-L_i-x')+\frac{1}{2}(L-L_i-x')(L+L_i+x')\right] \tag{27}$$

where the offset of each element from the rotational axis is given below.

$$L_i = (i-1)\frac{L}{N_e} \tag{28}$$

Here, L is the length of the whole beam and N_e is the nuber of elements, the beam is divided for the finite element formulation.

The finite element model of a rotating Timoshenko beam element that undergoes elongation and flapwise bending deformation is given in Fig.5. Here, it is seen that a two noded beam element that has eight degrees of freedom is preferred to model the beam. Here, w is the flapwise bending, θ is the angle due to flapwise bending and φ is shear angle which is the result of Timoshenko beam formulation.

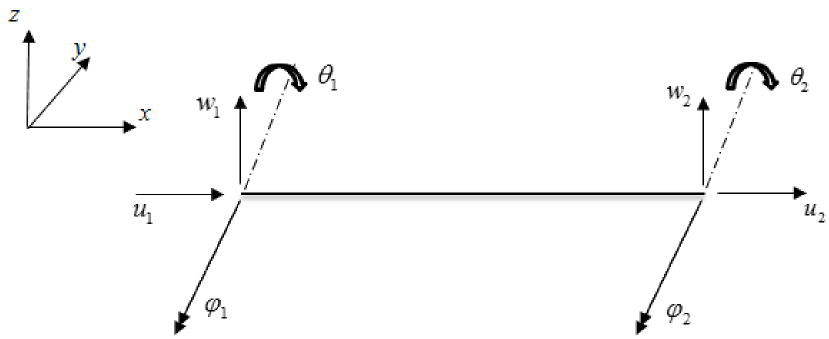

Figure 5. FEM of Timoshenko Beam

A Timoshenko beam element with two nodes per element and three degrees of freedom per node is considered.
Displacement fields of Timoshenko beam that undergoes flapwise bending deflection and extension are given by [Ozgumus, 2012] ;

$$u = a_0 + a_1 x \tag{29}$$

$$w = a_2 + a_3 x + a_4 x^2 + a_5 x^3 \tag{30}$$

$$\varphi = a_6 + a_7 x \tag{31}$$

$$\theta = w' - \varphi = a_3 - a_6 + (2a_4 - a_7)x + 3a_5 x^2 \tag{32}$$

Considering the displacement field polinomials given by Eqn. (29) – (32), the nodal displacements are determined as the displacement values at the first node of the element, x=0, and at the second node, x=L_e, respectively. These are given in matrix form as follows

$$\begin{bmatrix} u_1 \\ w_1 \\ \theta_1 \\ \varphi_1 \\ u_2 \\ w_2 \\ \theta_2 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & L_e & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L_e & L_e^2 & L_e^3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2L_e & 3L_e^2 & -1 & -L_e \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & L_e \end{bmatrix} \tag{33}$$

Here, $(\;)_1$ are the displacement values of the 1st node while $(\;)_2$ are the displacements on the 2nd node.

Relation between the displacement field and the nodal displacements is

$$\{q\} = [N]\{q_e\} \tag{34}$$

where for the present beam model, expressions of the displacements, $\{q\}$, the nodal displacements, $\{q_e\}$.

$$\{q\} = \{u \quad w \quad \theta \quad \varphi\}^T \tag{35}$$

$$[N] = \left[N_u \quad N_w \quad N_\theta \quad N_\varphi\right]^T \tag{36}$$

$$\{q_e\} = \{u_1 \quad w_1 \quad \theta_1 \quad \varphi_1 \quad u_2 \quad w_2 \quad \theta_2 \quad \varphi_2\}^T \tag{37}$$

where the matrix of the shape functions, $[N]$ are given by

$$[N_u] = \left\{1-\frac{x}{L} \quad 0 \quad 0 \quad 0 \quad \frac{x}{L} \quad 0 \quad 0 \quad 0\right\} \tag{38}$$

$$\begin{aligned}[N_w] = \Bigg\{0 \quad & 1-\frac{3x^2}{L_e^2}+\frac{2x^3}{L_e^3} \quad x-\frac{2x^2}{L_e}+\frac{x^3}{L_e^2} \quad x-\frac{2x^2}{L_e}+\frac{x^3}{L_e^2} \\ & 0 \quad \frac{3x^2}{L_e^2}-\frac{2x^3}{L_e^3} \quad -\frac{x^2}{L_e}+\frac{x^3}{L_e^2} \quad -\frac{x^2}{L_e}+\frac{x^3}{L_e^2}\Bigg\}\end{aligned} \tag{39}$$

$$\begin{aligned}[N_\theta] = \Bigg\{0 \quad & -\frac{6x}{L_e^2}+\frac{6x^2}{L_e^3} \quad 1-\frac{4x}{L_e}+\frac{3x^2}{L_e^2} \quad -\frac{3x}{L_e}+\frac{3x^2}{L_e^2} \quad 0 \\ & \frac{6x}{L_e^2}-\frac{6x^2}{L_e^3} \quad -\frac{2x}{L_e}+\frac{3x^2}{L_e^2} \quad -\frac{3x}{L_e}+\frac{3x^2}{L_e^2}\Bigg\}\end{aligned} \tag{40}$$

$$\left[N_\varphi\right] = \left\{0 \quad 0 \quad 0 \quad 1-\frac{x}{L} \quad 0 \quad 0 \quad 0 \quad \frac{x}{L}\right\} \tag{41}$$

Here $[N_u]$, $[N_w]$, $[N_\theta]$ and $[N_\varphi]$ are the shape functions associated with elongation, u, flapwise bending, w, angle due to flapwise bending, θ and shear angle, φ, respectively.

Considering the effect of centrifugal force and substituting the shape functions,i.e. Eqn. (38)-(41) into the potential and kinetic energy expressions, i.e. Eqn. (20) and Eqn.(24), the element stiffness matrice, $[K^e]$, and element mass matrice, $[M^e]$, are obtained as follows

$$\begin{aligned}\left[K^e\right] = \frac{1}{2}\int_0^{L_e} & (\bar{A}_{11}\left[\frac{dN_u}{dx}\right]^T\left[\frac{dN_u}{dx}\right] + 2\bar{B}_{11}\left[\frac{dN_u}{dx}\right]^T\left[\frac{dN_\theta}{dx}\right]\left[\frac{dN_\theta}{dx}\right]^T\left[\frac{dN_u}{dx}\right] \\ & +\bar{D}_{11}\left[\frac{dN_\theta}{dx}\right]^T\left[\frac{dN_\theta}{dx}\right] + \bar{A}_{55}\left(\left[\frac{dN_w}{dx}\right]-N_\theta\right)^T\left(\left[\frac{dN_w}{dx}\right]-N_\theta\right))dx\end{aligned} \tag{42}$$

$$\begin{aligned}\left[M^e\right] = \frac{1}{2}\int_0^{L_e} & (I_1[N_u]^T[N_u] + I_1[N_w]^T[N_w] + 2I_2[N_u]^T[N_u][N_\theta]^T[N_\theta] \\ & +I_3[N_\theta]^T[N_\theta])dx\end{aligned} \tag{43}$$

Here, the element stiffness matrix is derived from the potential energy expression and the element mass matrice is derived from the kinetic energy expression.

Depending on the number of elements used in the finite element modeling code, all the element matrices are assembled by considering the finite element rules to obtain the global matrices. The boundary conditions are applied to the global matrices to get the reduced matrices and the following matrix system of equations are obtained

$$[M]\{\ddot{q}\} + [K]\{q\} = \{0\} \quad (44)$$

where $[M]$ and $[K]$ are the reduced global mass and reduced global stiffness matrices, respectively.

Modal analysis is applied to Eqn.(34) to calculate the natural frequencies. Firstly, the modal matrix, $[\Phi]$, is calculated by using the eigenvectors obtained by solving the following determinant

$$\det\left[[K] - \omega^2 [M]\right] = 0 \quad (45)$$

Solving Eqn.(43), natural frequencies are calculated for the Euler Bernoulli and Timoshenko beam models.

RESULTS AND DISCUSSIONS

In this section, effects of several parameters, i.e. material distribution, rotational speed, slenderness ratio and the power law index, n on the vibration characteristics are examined for Timoshenko beams having different boundary conditions and material distribution properties.

The results are given in several tables and figures which is expected to be a very good source for the researchers who study in the field of functionally graded, rotating beams. When the results are compared with the ones in open literature, it is noticed that there is a very good agreement between the results which proves the correctness and accuracy of the studies in this paper.

To be able to make comparisons with the studies in open literature, the following dimensionless parameters given in Table 2 (Ozdemir O.,2016).

Table 2: Dimensionless Parameters

$\bar{x} = \dfrac{x}{L}$	$\bar{w} = \dfrac{w}{L}$	$\bar{u}_0 = \dfrac{u_0}{L}$
$\bar{\Omega}^2 = \dfrac{I_1 L^4 \Omega^2}{\bar{D}_{11}}$	$\lambda = \dfrac{\omega L^2}{h}\sqrt{\dfrac{\rho}{E}}$	$r^2 = \dfrac{I_3}{I_1 L^2}$

Here λ is the dimesionless frequency parameter, $\bar{\Omega}$ is the dimensionless angular speed parameter and r is the inverse of the slenderness ratio parameter.

Vibration characterics of rotating/nonrotating, functionally graded Timoshenko beams is examined for two different cases. In the first case, the beams have fixed-free end conditions while in the second case, the beam has simply-simply supported end conditions

The FG beam is made of Aluminum (Al) at the top and Alumina (Al_2O_3) at the bottom. The effective beam properties change through the beam thickness according to the power law. The material properties of the FG beam are displayed in Table 3.

Table 3: Material Properties of Functionally Graded Beam Model

Material Property	Aluminum (Al)	Alumina (Al_2O_3)
Modulus of Elasticity, E	70 GPa	380 GPa
Density, ρ	2702 kg/m³	3960 kg/m³
Poisson's Ratio, υ	0.3	0.3

Variation of the fundamental frequencies of a functionally graded, nonrotating Timoshenko beam with respect to the boundary conditions and power law index parameter is given in Table 4 for the slenderness ratio value of L/h=5. As mentioned in previous sections, the power law index parameter has a decreasing effect on the natural frequencies while the frequencies of Clamped-Free beams are higher than the frequencies of Simply-Simply Supported beams.

Table 4: Effect of Boundary Conditions and Power Law Index on The Natural Frequencies of A Nonrotating FG Timoshenko Beam (L/h=5)

Boundary Conditions	$\lambda = \frac{\omega L^2}{h}\sqrt{\frac{\rho_m}{E_m}}$	n=0	n=0.5	n=1	n=2	n=5	n=10
SS	**Present**	10.015	8.68448	7.91205	7.19908	6.65413	6.32658
	Şimşek (2010)	10.0705	8.74674	7.95034	7.17674	6.49349	6.16515
	Nguyen et al. (2015)	10.0726	8.7463	7.9518	7.1776	6.4929	6.1658
CF	**Present**	1.89482	1.61724	1.46304	1.33380	1.26449	1.2240
	Şimşek (2010)	1.89523	1.61817	1.46328	1.33254	1.25916	1.21834
	Nguyen et al. (2015)	1.8957	1.6182	1.4636	1.3328	1.2594	1.2187

Variation of the fundamental frequencies of a functionally graded, rotating Timoshenko beam with respect to power law index parameter and slenderness ratio is given in Table 5 for the dimensionless rotational speed of $\bar{\Omega}$=5.

Table 5: Effect of Slenderness Ratio and Power Law Index on The Natural Frequencies of A Rotating FG Timoshenko Beam with Clamped Free Boundary Conditions ($\bar{\Omega}$=5)

L/h	***n* =0**		***n* =0.2**		***n* =0.5**		***n* =1**	
	Özdemir (2019)	**Present**	**Özdemir (2019)**	**Present**	**Özdemir (2019)**	**Present**	**Özdemir (2019)**	**Present**
3	3.3483	3.3843	2.7451	2.8394	2.5883	2.7131	2.4172	2.5259
4	3.4358	3.4580	2.8159	2.9013	2.6592	2.7753	2.4888	2.5877
5	3.4834	3.4984	2.8536	2.9351	2.6968	2.8099	2.5273	2.6227

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