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AERODYNAMIC THRUST OPTIMIZATION OF A FLAPPING THIN AIRFOIL

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ABSTRACT

The thrust optimization of a flapping airfoil is studied with utilizing unsteady aerodynamic concepts based on Theodorsen's theory and Garrick's approach which is generalized for the pitching-plunging and deforming airfoils via rewriting the both thrust generated by lifting force and the leading edge suction velocity. The thrust function which is to be maximized is written in terms of the pitch, plunge and the varying camber amplitudes together with their constraints which make the amplitudes consistent with the theory utilized here. The gradient of the thrust function and the constraint is set to zero to achieve the maximum thrust condition as an eigenvalue problem. The maximum eigenvalue gives the maximum thrust and the corresponding eigenvector gives the pitch plunge and the camber amplitudes. Several cases of optimization are studied which include harmonic deformation, non-sinusoidal pitch plunge and the maximum thrust at zero freestream. The maximum thrust may give a poor result in terms of the aerodynamic efficiency. Here, the problem is reformulated so that the maximum thrust is obtained together with the desired efficiency or maximum efficiency. This requires solving the problem, in an iterative manner, with an additional constraint based on the desired efficiency or solving it with optimizing the multi-objective function.

INTRODUCTION

The studies of flapping wing aerodynamics have become quite popular since it provides quiet and efficient thrust, as opposed to the engine powered fixed wings, for the applications of MAV technology. Early 20th Century studies on flapping wing generated thrust were due to Knoller and Betz and in literature it is known as Knoller-Betz efffect which had been demonstrated first by [Katzmayer, 1922] experimentaly. By examining the wake of an oscillating airfoil one can decide if an unsteady thrust or drag is generated depending on the occurence of reverse or normal Karman vortex street is generated. [Jones et.al., 1998] showed the role of the reduced frequency and the plunge amplitude product, kh, on the thrust or drag generation both experimentally and numerically. Furthermore, [Gulcat, 2016] demonstrated numerically the effect of the Reynolds number, in addition to the kh value, on switching from drag to thrust of a heaving-plunging thin airfoil. The kh=0.2 is known as the critical value [Platzer, et al, 2008] at which zero drag or thrust is produced at certain Reynolds number. As the Reynolds number increases, thrust for lower values of kh is obtainable [Gulcat, 2009] in laminar fkows. The thrust generation is possible at zero free stream at low Reynolds numbers based on the angular frequency times the chord as the characteristic speed, $\mathrm{Re} = \omega c^2 / \nu$ <1000. This was shown experimentally by [Wang, 2000] at high angles of attack exceeding 20º.

Optimization of flapping airfoils for maximum thrust and effciency, based on Navier Stokes solutions, first appeared in [Tuncer and Kaya, 2005]. Furthermore, the nonsinusoidal path optimization of a priodically flapping airfoil was studied by implementing the nonuniform rational B-splines [Kaya and Tuncer, 2007]. Recently, the optimum thrust for a pitching-plunging airfoil at zero free stream is

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studied theoredically and experimentally in [Bulut et al, 2016], wherein the theoretical part is based on the Garrick's approach [Garrick, 1936] and its extended version in [Walker, 2012] and [Walker and Patil, 2014].

The objective of this study is to obtain the maximum thrust for a simple harmonically pitching-plunging and morfhing airfoil in finite and zero freestreams. The study is also extended to the path optimization for non-sinusoidal pitch-plunge motions, where doubling the thrust generation is possible compared to the results obtained by sinusoidal paths. The formulations are based on ideal flow therefore some constraints are applied on the amplitude of the motion to satisfy the ideal flow limitations. This way, the problem is solved as an eigenvalue problem which gives the maximum thrust as its maximum eigenvalue and the amplitude of the motion as the corresponding eigenvector. In addition, emposing an efficiency constraint results in with an iterative solution to achieve the maximum thrust and corresponding motion at that efficiency or iterating for the maximum efficiency as well.

METHOD

The method used here is based on the maximizitaion of the thrust force generated by the leading edge suction and the unsteady lifting force of an airfoil in pitch and plunge and camber deformations.
The leading edge suction force and the unsteady lift force contribute to the propulsive force generated by the flapping of a rigid airfoil is given as follows [Garrick, 1936]:

$$S = -(\pi \rho P^2 - \alpha L) \tag{1}$$

Where, $P = \lim_{x \to -1} (\gamma_a \sqrt{x^* + 1})/2$ is the leading edge suction velocity, α is the angle of attack and

$$\bar{L} = \rho U^2 b \left[-2C(k) \int_{-1}^{1} \sqrt{\frac{1+\xi^*}{1-\xi^*}} \frac{\overline{w}(\xi^*) d\xi^*}{U} - 2ik \int_{-1}^{1} \sqrt{1-\xi^{*2}} \frac{\overline{w}(\xi^*) d\xi^*}{U} \right] \tag{2}$$

is the lift amplitude, where $\overline{w} = i\omega \bar{z}_a + U \partial \bar{z}_a / \partial x$ is the amplitude of the downwash w obtained from the airfoil motion $z_a = z_a(x,t)$, and C(k)=F(k) + iG(k) is the Theodorsen function.

The amplitude of the vortex sheet strength, on the other hand, is given by, in terms of the circulation $\overline{\Gamma}_a$,

$$\bar{\gamma}_a(x^*) = \frac{2}{\pi} \sqrt{\frac{1-x^*}{1+x^*}} \int_{-1}^{1} \sqrt{\frac{1+\xi^*}{1-\xi^*}} \frac{\overline{w}(\xi^*) d\xi^*}{x^* - \xi^*} + \sqrt{\frac{1-x^*}{1+x^*}} \frac{ik\overline{\Gamma}_a}{b\pi} e^{ik} \int_{1}^{\infty} \sqrt{\frac{\lambda+1}{\lambda-1}} \frac{e^{-ik\lambda}}{x^* - \lambda} d\lambda \tag{3}$$

Where: $$\frac{ik\overline{\Gamma}_a}{b} e^{ik} = \frac{4/\pi}{H_1^{(2)}(k) + H_0^{(2)}(k)} \int_{-1}^{1} \sqrt{\frac{1+\xi^*}{1-\xi^*}} \overline{w}(\xi^*) d\xi^*$$

The integral in the second term of the right hand side can be expanded into the series in terms of $\frac{x}{\lambda}$ as follows

$$I(k,x) = \int_{1}^{\infty} \sqrt{\frac{\lambda+1}{\lambda-1}} \frac{e^{-ik\lambda}}{x-\lambda} d\lambda = -\int_{1}^{\infty} \frac{\lambda+1}{\sqrt{\lambda^2-1}} \frac{1}{\lambda} (1 + \frac{x}{\lambda} + \frac{x^2}{\lambda^2} + \frac{x^3}{\lambda^3} + \ldots) e^{-ik\lambda} d\lambda$$

$$= -\int_{1}^{\infty} \frac{1}{\sqrt{\lambda^2-1}} \left[1 + \frac{1}{\lambda}(x+1) + \frac{1}{\lambda^2}(x^2+x) + \frac{1}{\lambda^3}(x^3+x^2) + \ldots \right] e^{-ik\lambda} d\lambda, \; -1 < x < 1$$

$$= i\frac{\pi}{2} H_0^{(2)}(k) - \int_{1}^{\infty} \frac{1}{\sqrt{\lambda^2-1}} \left[\frac{1}{\lambda}(x+1) + \frac{1}{\lambda^2}(x^2+x) + \frac{1}{\lambda^3}(x^3+x^2) + \ldots \right] e^{-ik\lambda} d\lambda$$

As x approaches -1, the second term vanishes to leave us with

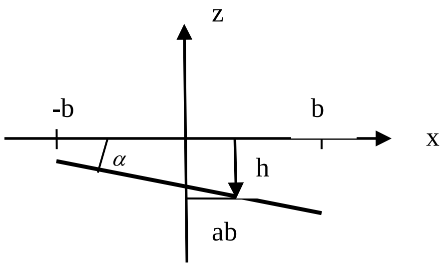
$$\overline{P}_i = \frac{\sqrt{2}}{\pi}\int_{-1}^{1}\sqrt{\frac{1+\xi^*}{1-\xi^*}}\frac{\overline{w}_i(\xi^*)\,d\xi^*}{-1-\xi^*} + \frac{\sqrt{2}}{\pi}\left[1-C(k)\right]\int_{-1}^{1}\sqrt{\frac{1+\xi^*}{1-\xi^*}}\overline{w}_i(\xi^*)\,d\xi$$

i) For a heaving-plunging thin airfoil in SHM $z_a(x,t) = -he^{i\omega t} \;\; with \;\; \overline{w} = -hi\omega$, Hence,

$$\overline{P}_0 = \sqrt{2}C(k)ik\overline{h}U, \;\; \overline{h} = h/b$$

and $$\overline{L}_0 = \rho U^2 b\left[2C(k)ik - k^2\right]\pi\overline{h}$$

ii) For pitching plunging airfoil with pitch axis at ab :



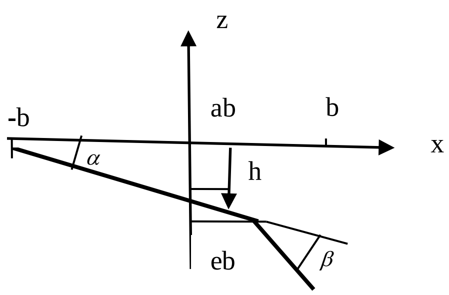
$$z_a(x,t) = -\overline{\alpha}\,e^{i\omega t}(x-ab) \; with \;\; \overline{w}(x) = -\overline{\alpha}\,i\omega(x-ab) - U\overline{\alpha}$$

The leading edge suction velocity then becames

$$\overline{P}_1 = \sqrt{2}\left[C(k) + (C(k)-1)ik/2 + (1-C(k))a - a\,ik\right]U\overline{\alpha}$$

and the lift $\overline{L}_1 = \rho U^2 b\left[2C(k)(ik/2+1) + ik - 2C(k)ika - ika + k^2 a\right]\pi\overline{\alpha}$

iii) For oscillating flap



$$z_a(x,t) = -\overline{\beta}\,e^{i\omega t}(x-eb) \; with \;\; \overline{w}(x) = -\overline{\beta}\,i\omega(x-eb), \;\; for \; x \geq eb$$

The leading edge suction velocity and the lift due to flap oscillations then read as [Garrick, 1936]

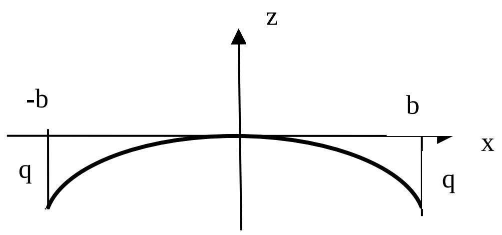
$$\overline{P}_{1\beta} = \sqrt{2}(-2U\sqrt{1-e^2}\,\beta + T_4 b\dot{\beta})/(2\pi) \quad \text{and,}$$

$$\overline{L}_{1\beta} = \rho U b^2(UT_4\dot{\beta} + bT_1\ddot{\beta}) + 2\pi\rho UbC(k)(UT_{10}\beta + bT_{11}\dot{\beta}/2)/\pi$$

Where,
$$T_1 = -(2+e^2)\sqrt{1-e^2}\;/2 + e\,\cos^{-} e, \quad T_4 = e\sqrt{1-e^2}\; - \cos^{-} e$$
$$T_{10} = \sqrt{1-e^2}\; + \cos^{-} e, \quad T_{11} = (2+e)\sqrt{1-e^2}\; + (1-2e)\cos^{-} e$$

iv) For parabolic camber oscillations:

$$z_a(x,t) = -q x^2 / b^2 e^{i\omega t} \;\; with \; \overline{w} = -(i\omega x^2/b^2 + 2x/b^2 U)\overline{q}$$



The leading edge suction force reads as

$$\overline{P}_2 = \sqrt{2}\left[C(k)ik/2 + C(k) - 1\right]Uq/b$$

and the lift contribution reads as

$$(\overline{\alpha}\,\overline{L}_2) = \int_{-b}^{b} (p_L - p_U)\frac{dz_A}{dx}dx$$

Using the expression for the lifting pressure the above integral reads as

$$\frac{\overline{\alpha}\,\overline{L}_2}{\rho U^2 b} = -3\pi\, ik\, q^{*2} + 4\pi(1-C)(-1-ik/2)q^{*2} + \pi k^2 q^{*2}$$

Now we can write down a general expression for the thrust generated by a having-plunging, pitching and morphing thin airfoil. That is

$$T = T_{LES} + \int_{-b}^{b} (p_L - p_U)\frac{dz_a}{dx}dx$$

$$T_{LES} = \rho\pi U^2 b\left(\sum_{i=0}^{2} P_i\right)\left(\sum_{j=0}^{2} P_j\right)$$

and

$$\sum_{i=0}^{2} \alpha \overline{L}_i = \sum_{i=0}^{2} \int_{-b}^{b} (\overline{p}_{Li} - \overline{p}_{U_i})\frac{dz_{ai}}{dx}dx = -\overline{\alpha}\overline{L}_0 - \overline{\alpha}\overline{L}_1 - \overline{\alpha}\,\overline{L}_2$$

Now, we can form a table showing the sectional suction and lift coefficients as follows:

Table1: Sectional suction and lift coefficients

	$\overline{P}$	$\overline{\alpha}\,\overline{L}$
$\overline{h}$	$\sqrt{2}C(k)ik$	$2C(k)ik - k^2$
$\overline{\alpha}$	$\sqrt{2}(C(k) + (C(k)-1)ik/2)$	$2C(k)(ik/2+1) + ik$
	$+\sqrt{2}((1-C(k))a - a\,ik)$	$-2C(k)ika - ika + k^2 a$
$\overline{q}$	$\sqrt{2}(C(k)ik/2 + C(k) - 1)$	$ik + (1-C)(-1-ik/2) + k^2$
$\overline{\beta}$	$\sqrt{2}(-2\sqrt{1-e^2} + T_4 ik)/2\pi$	$T_4 ik - T_1 k^2 + 2(T_{10} + T_{11} ik/2)/\pi$

Here, T_i are listed in [Garrick, 1936], Let us form a vector **Q** with elements $\vec{Q} = \left| \bar{h}\,\bar{\alpha}\,\bar{q}\,\bar{\beta} \right|$, and consider h=$h_r$ as real and $\bar{\alpha} = \alpha_r + i\alpha_i$ and $\bar{q} = q_r + iq_i$ and $\bar{\beta} = \beta_r + i\beta_i$ are complex to indicate that these are out of phase with h. Hence we have the complex quantities represented as a real vector

$$\bar{Q} = \begin{Bmatrix} h_r \\ \alpha_r \\ \alpha_i \\ q_r \\ q_i \end{Bmatrix} \quad \text{or} \quad \bar{Q} = \begin{Bmatrix} h_r \\ \alpha_r \\ \alpha_i \\ \beta_r \\ \beta_i \end{Bmatrix}$$

This enables us to write a quadratic form for the average thrust as follows

$$T_{avg} = \frac{1}{2}\mathrm{Re}\left[\pi\rho\bar{P}\,conj(\bar{P}) + \sum_{i=0}^{4} \bar{\alpha}\,\bar{L}_i \right] \tag{4}$$

Hence, only the real part of the the equation above contributes to the average thrust. The imaginary part contains product of sine and cosine which integrates to zero over one period to give

$$T_{avg} = \frac{1}{2}\pi\rho U^2 b \{Q\}^T \left[H_{avg} \right] \{Q\} \tag{5}$$

Wherein, H_{avg} is a 5x5 matrix which constitutes the quadratic form for the thrust function which has all real entries as functions of the reduced frequency k and the real and imaginary parts of the Theodorsen function, i.e. F and G.

Optimization

The average thrust function given with (5) can be maximized if its gradient is set equal to zero, i.e.

$$\bar{\nabla} T_{avg} = \vec{0} \quad \rightarrow \quad \pi\rho U^2 b \left[H_{avg} \right] \{Q\}_{\max T} = \vec{0}$$

Where, $\{Q\}_{\max T}$ is the vector which gives the maximum value of the average thrust. The solution of this equation is trivial since the H_{avg} is not singular. On the other hand, the derivation of the equation is based on the linear aerodynamic theory, therefore, any increase in the elements of Q causes increase in thrust. Hence, there is not any maximum as the problem posed. However, we can find a maximum thrust if we impose restrictions on the motion as constraint. There are several ways to impose the constraint for the optimization; one of them is the magnitude constraint.

Magnitude constraint: Both aerodynamic and the mechanical restrictions can justify the magnitude constraint in the following form

$$\sum_{i=1}^{5} Q_i^2 \le 1 \qquad \text{or} \qquad \{Q\}^T \{Q\} \le 1$$

(6)

With this restriction, we stop the increase in maximum thrust with setting a limit to the magnitude of the motion. In the design space this constraint will put a peak to the maximum thrust. Hence, we can write the constraint as an equality, i.e.

$$f(Q) = \{Q\}^T \{Q\} - 1$$

This enables us to write a Lagrangian composed of the average thrust function and the constraint without altering the value of the average thrust to be maximized as follows:

$$L(Q,\lambda) = \frac{1}{2}\pi\rho U^2 b \{Q\}^T \left[H_{avg} \right] \{Q\} - \lambda\, f(Q) \tag{7-a}$$

Wherein, λ is the Lagrange multiplier for the constraint. Now, we can set the gradient of the Lagrangian (7) to zero to obtain

$$\pi\rho U^2 b\left[H_{avg}\right]\{Q\} - \lambda\{Q\} = 0 \quad \text{(7-b)}$$

$$\{Q\}^T\{Q\} - 1 = 0$$

The first equation is nothing but an eigenvalue problem. It has *n* principle directions and corresponding *n* eigenvalues in general. The largest eigenvalue is the maximum average thrust and the corresponding eigenvector gives the associated motion and the deformation vector. With the aid of the second equation we can prove that the Lagrangian is the maximum. If we rewrite the Lagrangian with $\{Q\}_{\max}{}^T\{Q\}_{\max} - 1 = 0$ we then obtain

$$L(Q_{\max},\lambda_{\max}) = \frac{1}{2}\pi\rho U^2 b\{Q\}_{\max}{}^T\lambda_{\max}\{Q\}_{\max} = \frac{1}{2}\pi\rho U^2 b\lambda_{\max}$$

Example:

Let us consider a thin airfoil undergoing plunge-pitch and morfhing simultaneously at k=0.5. Let us find the maximum value of the aerodynamic thrust and the corresponding motion and the deformation.

Using above the formulation given, with pitch point changing, we find the values given in the following Table 2, where $T_{\max} = L(Q_{\max},\lambda_{\max})/(\rho U^2 b)$

Table2: Maximum average thrust for various pitch points.

a	T_{max}	h^*	$\overline{\alpha}_r$	$\overline{\alpha}_i$	q_r	q_i
-1.0	1.9578	0.9368	0.1379	-0.2986	-0.0609	-0.1025
-0.5	1.5298	0.9577	0.1093	-0.2485	-0.0491	-0.0823
0.0	1.2505	0.9656	0.0804	-0.2380	-0.0325	-0.0590

As seen from the Table 2, the minimum of the maximum thrust occurs for the pitch point located at midchord. The pitch point location towards the leading edge increases the value of the thrust within the limits of the linear theory. The choice of pitch point location away from the midchord towards the trailing edge increases the maximum thrust but the motion associated with exceeds the limits of linear theory.

Efficiency constraint

The thrust optimization with magnitude constraint does not necessarily result in most efficient solution. Therefore, we need to consider the optimization aiming a desired efficiency. For this purpose, the efficiency can be forced to a certain value while the thrust is made maximum. Hence, generating a Pareto front for thrust and efficiency is the proper way of optimizing the thrust with efficiency constraint. (On the other hand, the problem of efficiency constraint without a magnitude constraint is not a well posed problem and is not considered.) The thrust efficiency is defined as the ratio of the work generated by thrust to the work required for the motion of the airfoil. Thus it is

$$\eta = \frac{W_T}{W_M} = \frac{T_{avg}U}{W_M} \quad (8)$$

The work required for the motion of the airfoil for one period reads as

$$W_M = \frac{1}{2}\rho U^2 \int_0^T\int_{-b}^{b} C_{pa}(x,t)\frac{\partial z_a(x,t)}{\partial t}dx\,dt\,, \text{ where } \frac{\partial z_a}{\partial t} = i\omega\,\overline{z}_a(x)e^{i\omega t} \quad and \quad C_{pa} = \overline{C}_{pa}e^{i\omega t}$$

With proper non-dimensionalization of the integral the average of this work becomes

$$W_{Mav} = \rho U^3 b\frac{1}{2}\mathrm{Re}\left(\int_{-1}^{1}\overline{C}_{pa}ik\,\overline{z}_a^*dx^*\right)$$

Substituting the motion and the deformation parameters of the airfoil, this result can be written in matrix form as follows

$$W_{Mav} = \frac{1}{2}\pi\rho U^3 b\{Q\}^T \left[H_{avg}\right]\{Q\}$$

The efficiency constraint can be written in terms of chosen reference efficiency as follows

$$\eta(\{Q\},k) - \eta_{ref} = 0$$

Then we have

$$\eta(Q\ ) = \{Q\}^T \left\langle \left[H_{avg}\right] - \eta_{ref}\left[H_M\right] \right\rangle \{Q\} = 0 \quad (9)$$

This enables us to write the Lagrangian

$$L(Q\ ,\lambda,\lambda_\eta) = \frac{1}{2}\pi\rho U^2 b\{Q\}^T \left[H_{avg}\right]\{Q\} - \lambda f(Q\ ) - \lambda_\eta \eta(Q\ ) \quad (10)$$

Here, λ_η is the Lagrange multiplier for the efficiency constraint. Now, we can find the maximum of it by setting the gradient of the Lagrangian [10] to zero vector to get

$$\begin{aligned} &\pi\rho U^2 b\left[H_{avg}\ \right]\{Q\} - \lambda\{Q\} - \lambda_\eta \left\langle \left[H_{avg}\right] - \eta_{ref}\left[H_M\right] \right\rangle \{Q\} = 0 \\ &\{Q\}^T\{Q\} - 1 = 0 \\ &\{Q\}^T \left\langle \left[H_{avg}\right] - \eta_{ref}\left[H_M\right] \right\rangle \{Q\} = 0 \end{aligned} \quad \text{(11-a,b,c)}$$

This is no longer an eigenvalue problem. It involves N unknowns for the motion and the deformation amplitudes and also 2 Lagrange multipliers to result in to solve N+2 nonlinear equations. For the solution we use an iterative method for systems. Sole efficiency constraint problem without a magnitude constraint looks like a generalized eigenvalue problem, however, it turns out to be an ill-posed problem, and therefore it is not looked into. For any given efficiency value, we can solve the double eigenvalue problem with an iterative procedure as described below. (On the other hand, the solution for the maximum thrust case provides a reasonably good estimate to start the iterations for the numerical method to create a specific front for the optimization.)

Let $f(x_i)=0$ be the non-linear system of equations to be solved iteratively. Expanding f_i about x_i and setting it to zero, in indicial notation, gives

$$f_{i+1} = f_i + \frac{\partial f_i}{\partial x_j}\delta x_j \cong 0 \quad \rightarrow \quad \frac{\partial f_i}{\partial x_j}\delta x_j = -f_i$$

For each iteration, the following matrix equation is solved for δx_j

$$\sum_j c_{ij}\delta x_j = -f_i \quad with \quad c_{ij} = \frac{\partial f_i}{\partial x_j}$$

Hence, $x_i^{new} = x_i^{old} + \delta x_i$ until convergence!

The elements of the matrix c_{ij} and the load vector f_i are given in the appendix of the paper.

As an application to the efficiency constraint the pitching-plunging airfoil with k=0.5 having various efficiencies the following results from (11) are obtained and provided at Table 3.

Table3: Optimum Values obtained with efficiency constraints η for k=0.5

η	T_{max}	h^*	$\overline{\alpha}_r$	$\overline{\alpha}_i$	λ_1	λ_2
0.30	0.2127	0.9752	0.2111	0.0663	0.2127	0.1795
0.40	0.2139	0.9760	0.2074	-0.0661	0.2139	-0.1332
0.50	0.2031	0.9677	0.1958	-0.1590	0.2031	-0.3174
0.60	0.1902	0.9592	0.1770	-0.2219	0.1901	-0.4014
0.70	0.1786	0.9520	0.1591	-0.2648	0.1783	-0.4263
0.11	0.2158	0.9776	0.2104	-0.0085	(no efficiency constraint)	
0.77	0.1397	0.9612	0.2174	-0.1699	(maximum efficiency)	

The convergence is quite fast. Above results are obtained with 3 decimal place accuracy in 4 iterations and with 6th decimal place accuracy in 10 iterations. If one compares the results for the efficiencies for 0.30 and 0.60, one finds that the efficiency is doubled whereas the maximum thrust is reduced only 10%. The last row of Table 3 shows the results of the case for which only the magnitude constraint is considered, wherein T_{max}=0.2158 is found as the highest eigenvalue with the lowest efficiency.

Efficiency optimization

Using multi-objective optimization, it is possible to optimize the motion to yield the maximum efficiency. The multi-objective optimization requires an iterative procedure, because of nonlinear dependence of the efficiency on the motion. based on the creation of a Pareto front while maximizing the average thrust. The previously defined efficiency in terms of the motion *Q* reads

$$\eta = \frac{W_T}{W_M} = \frac{\{Q\}^T\left[H_{avg}\right]\{Q\}}{\{Q\}^T\left[H_M\right]\{Q\}}$$

Now, we want to maximize the efficiency as well as the average thrust function with magnitude constraint. This requires maximization of (7-a) which is $L_1(Q,\lambda)$ and $L_2(Q,\eta)$ which is the efficiency. The maximization of the efficiency requires $\partial L_2(Q,\eta)/\partial Q = 0$. Hence, we get

$$\frac{W_M\left[H_{avg}\right]\{Q\} - W_T\left[H_M\right]\{Q\}}{W_M^2} = 0$$

Combining maximization of L_1 and L_2 yields

$$\pi\rho U^2 b\left\langle \left[H_{avg}\right] + W_M(\{Q\})\left[H_{avg}\right]\{Q\} - W_T(\{Q\})\left[H_M\right]\right\rangle\{Q\} - \lambda\{Q\} = 0$$

as a new eigenvalue problem which is nonlinear due to presence of W_M ($\{Q\}$) and W_T ($\{Q\}$). These new set of equations converge to 4 decimal places in 4 iterations to give the last row of Table 3. For this case, efficiency is the highest but the average thrust is the lowest as given in Table 3.

Zero freestream

The thrust generated at zero freestream represents the case which corresponds to the motion starting from the rest. The reduced frequency at zero free stream becomes infinity, therefore, the Theodorsen Function $C(\infty) = 0.5$. This gives us the Table 4 as follows

Table4: Suction and lift coefficients for U=0

	$\overline{S}$	$\overline{\alpha}\,\overline{L}$
$\overline{h}$	$\sqrt{2}Fi\omega b$	$-\omega^2 b^2$
$\overline{\alpha}$	$\sqrt{2}((F-1)/2-a)i\omega b$	$\omega^2 b^2 a$
$\overline{q}$	$\sqrt{2}Fi\omega b/2$	$\omega^2 b^2$
$\overline{\beta}$	$\sqrt{2}T_4 i\omega b/2\pi$	$-T_1\omega^2 b^2$

Figure 1. shows the various positions of the deforming thin airfoil while pitching and plunging with maximum thrust.

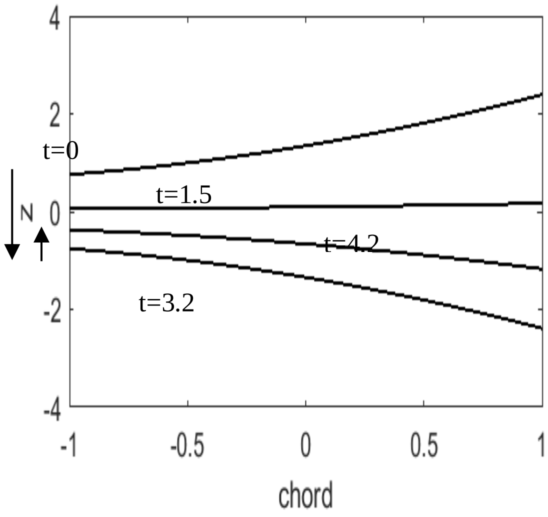


Figure 1: Motion & Deformation of thin airfoil for zero free stream at a=-0.5 and $\omega = 1$,

(plunging for 0<t<3.2 and heaving for 3.2<t<4.2)

Non-sinusoidal path optimization

We can have an airfoil which may not necessarily flap simple harmonically to produce high aerodynamic thrust. For that purpose we can employ non-uniform rational B splines (NURBS) like described and implemented in [Kaya and Tuncer, 2007]. Let in two dimensions S(u)=[x(u), y(u)] be the smooth curve described with the following parametric representation

$$x(u) = \frac{2P_1 u(1-u)^2 + 2P_2 u^2(1-u)}{(1-u)^3 + u(1-u)^2 + u^2(1-u) + u^3}$$

$$y(u) = \frac{-(1-u)^3 - u(1-u)^2 + u^2(1-u) + u^3}{(1-u)^3 + u(1-u)^2 + u^2(1-u) + u^3}$$

The non-sinusoidal periodic function is then defined as

$$\text{f}[\text{u}(\omega t)] = \text{y(u)} = \text{f}(\omega t) \quad \text{where} \quad \tan(\omega t) = -\frac{x(u)}{y(u) - P_0}$$

For a known ωt the above equation is solved for u to determine y(u)=f(ωt). Here, x and y determine a closed curve where P_0 defines the center and P_1 and P_2 indicates the flatness of the NURBS curve. Once the NURBS curve are found, the pitch-plunge motion is described as

$$h = -\bar{h}\ f_h(\omega t), \quad \alpha = -\bar{\alpha}\ f_\alpha(\omega t + \phi) \tag{12}$$

Where, ω is the angular frequency, $k = \omega b / U$ is the reduced frequency and ϕ is the phase difference with pitch and plunge.

The thrust optimized results are here, borrowed from [Kaya and Tuncer, 2007], for k=0.5 and h=0.5 reads as $\bar{\alpha} = 21.2^0$, $\phi = 41.4^0$, P_{0h}=-0.9, P_{1h}= P_{2h}=3.5, $P_{0\alpha} = -0.8$, $P_{1\alpha} = P_{2\alpha} = 0.2$.

Shown in Figure 2 is the periodic path of the pitch and plunge obtained with above parameters.

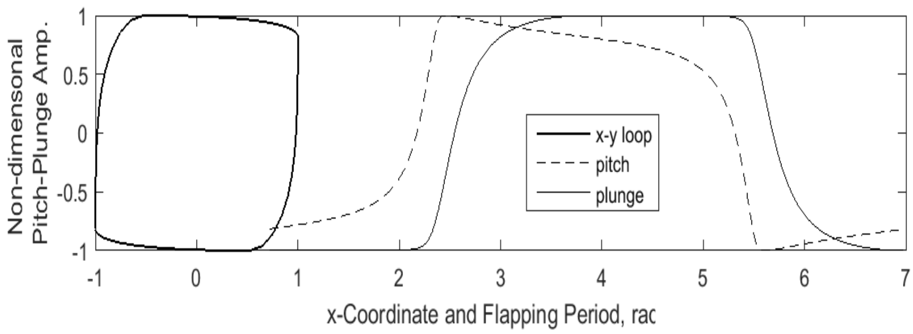


Figure 2: Normalized x-y loop for plunge (left) and pitch-plunge amplitude variation for a period.

Based on the Wagner function [BAH], now, we can evaluate the thrust induced by the nonsinusoidal motion as follows. The contribution of the lift to the thrust can be found directly by means of Wagner function. The leading edge suction's contribution, however, requires extra considerations. The leading edge suction velocity for a pitching-plunging motion of an airfoil with the aid of Table.1 (for a=0) is given as

$$P=\sqrt{2}C(k)\dot{h}+\sqrt{2}C(k)U\alpha+\sqrt{2}(C(k)-1)\dot{\alpha}\,b/2 \tag{13}$$

If we collect the coefficients for the motion together we obtain

$$P=\sqrt{2}C(k)(\dot{h}+U\alpha+\dot{\alpha}\,b/2)-\dot{\alpha}\,b/2$$

The term in the parenthesis is nothing but the downwash of the arbitrary motion at the quarter chord of the airfoil. Hence,

$$P=\sqrt{2}C(k)\,w(t,b/2)-\dot{\alpha}\,b/2 \tag{14}$$

With the aid of Fourier transform the first term of *P* can be written for all the frequencies involved in the arbitrary non-sinusoidal unit change w_0 in downwash as follows

$$P(s)=w_0\int_{-\infty}^{\infty}\frac{C(k)}{ik}e^{iks}dk-\dot{\alpha}\,b/2, \qquad k=\omega b/U \;and\; s=Ut/b$$

The improper complex integral is related to the well known Wagner function $\varphi(s)$ so that

$$P(s)=\sqrt{2}\;\; w_0\,\varphi(s)-\dot{\alpha}\,b/2 \qquad \text{with} \qquad \varphi(s)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{C(k)}{ik}e^{iks}dk=1-0.165e^{-0.0455s}-0.335e^{-0.3s}$$

Here, the Wagner function is the indicial admittance for unit excitation. For the arbitrary downwash w(t,b/2) the leading edge suction velocity becomes

$$P(s)=\sqrt{2}\left[\;w(b/2,s)\,\varphi(0)+\int_0^s w(b/2,\sigma)\,\varphi'(s-\sigma)\,d\sigma\right]-\dot{\alpha}\,b/2 \tag{15}$$

This is very similar to the lift and moment generated by the arbitrary motion which read as

$$L(s)=\pi\rho b\left[\vec{h}+U\dot{\alpha}\right]-2\rho Ub\pi\left[\;w(b/2,s)\,\varphi(0)+\int_0^s w(b/2,\sigma)\,\varphi'(s-\sigma)\,d\sigma\right] \tag{16}$$

$$M(s)=-\pi\rho b^2\left[b\ddot{\alpha}/4+U\dot{\alpha}\right]/2-\rho Ub\pi\left[\;w(b/2,s)\,\varphi(0)+\int_0^s w(b/2,\sigma)\,\varphi'(s-\sigma)\,d\sigma\right] \tag{17}$$

The total thrust due the leading edge suction and the lift is found from

$$S = -(\pi \rho b P^2 - \alpha L)$$

and plotted on the Figure 3.

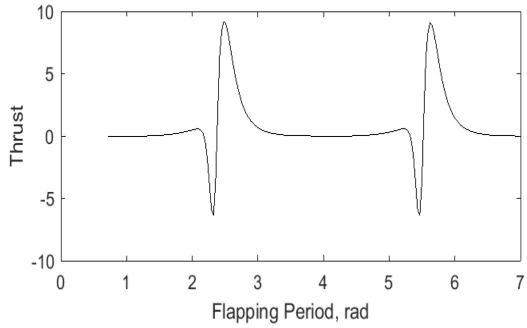


Figure 3: Thrust variation for a period

The time averaged unsteady thrust coefficient, integrated over one period of time gives the following thrust average thrust coefficient $C_T = 0.72$ as opposed to the sinusoidally generated optimum thrust coefficient $C_t = 0.31$.

The aerodynamic efficiency is defined before for the sinusoidal motions. For the non-sinusoidal pitch-plunge the the work done by the motion is the product of the force with the vertical velocity over a period of motion, hence

$$W_M = \frac{1}{2}\rho U^2 \int_0^T \int_{-b}^{b} C_{pa}(x,t)\frac{\partial z_a(x,t)}{\partial t}\,dx\,dt = \frac{1}{2}\rho U^2 \int_0^T \int_{-b}^{b} C_{pa}(x,t)(-\dot{h} - U\alpha - U\dot{\alpha}x)\,dx\,dt$$

This equation is simplified with performing the integral over the chord to get

$$W_M = \frac{1}{2}\rho U^2 \int_0^T C_L(-\dot{h} - U\alpha)\ dt + \frac{1}{2}\rho U^2 \int_0^T C_M(-U\dot{\alpha})\ dt \tag{18}$$

We can, now, calculate the value of the efficiency for the above example using the Wagner function to evaluate C_L and C_M from (16) and (17). Thus, the numerical value of the efficiency becomes

$$\eta = \frac{W_T}{W_M} = 0.28 \tag{19}$$

Thrust at Zero Free-Stream: Non-sinusoidal optimum solutions at zero free stream can be obtained with setting U=0 for the leading edge suction velocity P and the sectional lift L from (15) and (16):

$$P(s) = \sqrt{2}\left[\ \dot{h}(s)\varphi(0) + \int_0^s \dot{h}(\sigma)\,\varphi'(s-\sigma)\,d\sigma\right] - \dot{\alpha}\,b/2 \quad \text{and} \quad L(s) = \pi\,\rho\,b\,\vec{h} \tag{20-a,b}$$

Shown in Figure 4 is the normalized thrust coefficient variation in one period of flapping.

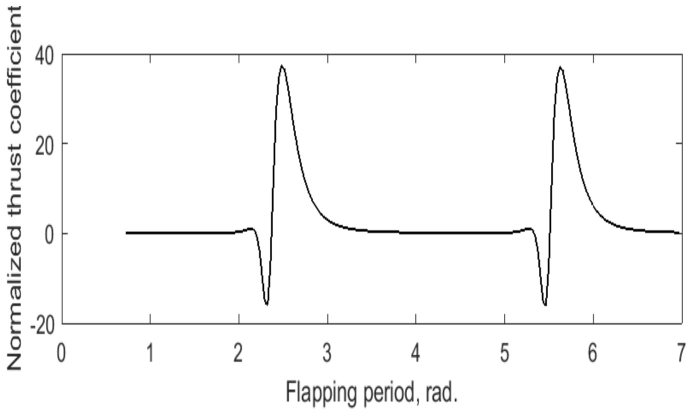


Figure 4: Normalized thrust, $T/(\rho\omega^2 b^3)$, variation for one period of flapping.

The averaged thrust coefficient from Figure 4 reads as 1.74 which is much higher than the thrust coefficient obtained with the sinusoidal flapping.

Above optimization is not based on any constraint. Next, we introduce the optimization with magnitude constraint.

Magnitude Constraint: The thrust function *S* can be optimized, using *P* and *L* expressions given above, for the non-sinusoidal flapping motion as an eigenvalue problem as follows.

$$\begin{bmatrix} P_h^2 & -L_h \\ -L_h & P_\alpha^2 - L_\alpha \end{bmatrix} \begin{pmatrix} h \\ \alpha \end{pmatrix} = \lambda \begin{pmatrix} h \\ \alpha \end{pmatrix} \tag{21}$$

The eigenvalues of (21) read as $\lambda_{\min}$ = -0.7912 and $\lambda_{\max}$ = 0.7869, and the corresponding eigenvectors are x_{min}= (0.2646, 0.9644) and x_{max}= (-0.9644, 0.2646). Hence, example problem solved with the pitch-plunge motion constraint as described before gives the maximum thrust coefficient as C_T=0.78 with

$$z_a(x,t) = 0.96\, f_h(\omega t) - 0.26x f_\alpha(\omega t + \phi) \tag{22}$$

Shown in Figure 5 is the thrust variation based on the magnitude constraint.

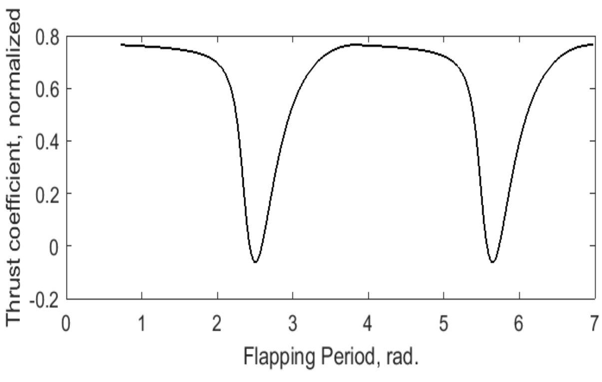


Figure 5: Normalized thrust variation for the optimization with magnitude constraint

Power Extraction: By means of a periodic motion, sinusoidal or non-sinusoidal, it is possible to obtain negative thrust which implies power extraction from the airfoil. The optimum power extraction with non-sinusoidal oscillations is obtained with magnitude constraint as follows:

$$z_a(x,t) = -0.26\, f_h(\omega t) - 0.96\, x\, f_\alpha(\omega t + \phi) \tag{23}$$

The drag coefficient for this case reads as C_D*=-0.79.* The variation of the drag coefficient is shown in Figure 6.

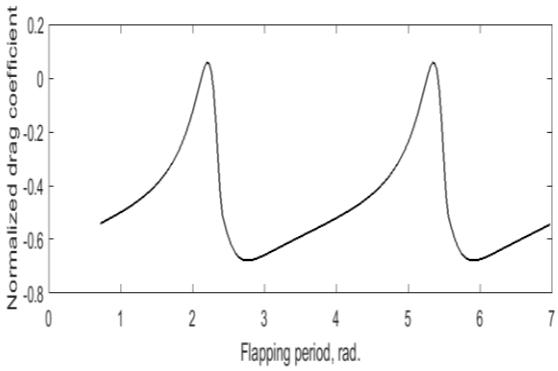


Figure 6: Normalized drag coefficient variation for the non-sinusoidal power extraction.

CONCLUSIONS

The aerodynamic thrust optimization of a thin airfoil is made for both simple harmonic and non-sinusoidal paths. Pitch, plunge and chord wise morphing of the airfoil are considered. The motion on-set from the rest is also studied under the thrust optimization at zero free stream. Finally, the maximum power extraction via flapping is studied. The following facts are observed:

i) Pitch point location has an effect on the magnitude of the thrust; the thrust increases as the pitch point moves from mid-chord towards the leading edge,

ii) the magnitude constraint renders the problem to an eigenvalue problem while maximizing the Lagrangian,

iii) comparison with the experimental study is satisfactory, especially at high reduced frequencies,

iv) efficiency constraint is a good tool to increase the performance with small loss of thrust amplitude,

v) imposing the efficiency constraint makes the problem solution in an iterative way which may yield to non-converging solutions if high efficiencies are sought,

vi) maximization of the efficiency results in low average thrust value,

vii) non-sinusoidal path optimization gives much more thrust compared to the sinusoidal path optimizations,

viii) maximum thrust occurs at high plunge low pitch amplitudes, whereas maximum power extraction occurs at low plunge and high pitch amplitudes.

Appendix

The elements of H_{avg} and H_m matrices are

H_{avg}(1,1)=2*(F*F+G*G)*k*k;
H_{avg} (1,2)=2*k*F*F-k*F-G*k*k+2*G*G*k;
H_{avg}(1,3)=(F*F+G*G)*k*k-F*k*k+k*k/2+G*k;
H_{avg}(2,2)=0.5*(F*F+G*G)*k*k+2*F*F-k*k*F-2*F+2*G*G+k*k/2-G*k; H_{avg}(3,3)= H_{avg}(2,2);
H_{avg}(2,1)= H_{avg}(1,2); H_{avg}(3,1)= H_{avg}(1,3);

and,

H_m(1,1)=4*F*k*k; H_m(1,2)=2*G*k+k*k; H_m(1,3)=2*F*k-2*G*k*k;
H_m(2,1)= H_m(1,2); H_m(3,1)= H_m(1,3);
H_m(2,2)=-2*G*k+k*k-F*k*k; H_m(3,3)= H_m(2,2);

Now, the set of equations were

$$\pi\rho U^2 b\left[H_{avg}\right]\{Q\}-\lambda\{Q\}-\lambda_\eta\left\langle\left[H_{avg}\right]-\eta_{ref}\left[H_M\right]\right\rangle\{Q\}=0$$

$$\{Q\}^T\{Q\}-1=0$$

$$\{Q\}^T\left\langle\left[H_{avg}\right]-\eta_{ref}\left[H_M\right]\right\rangle\{Q\}=0$$

Let a_{ij} *be the elements of* $\left\lfloor H_{avg}\right\rfloor$ and b_{ij} *be the elements of* $\left\langle\left[H_{avg}\right]-\eta_{ref}\left[H_M\right]\right\rangle$

Setting the unknown vector ***x = Q,*** then the equations in indicial notain (repeated index implies summation) read as

$$(a_{ij}-\lambda\delta_{ij}-\lambda_\eta b_{ij})x_j=0$$
$$x_j\,x_j-1=0$$
$$x_i\,b_{ij}\,x_j=0$$

Wherein, $\lambda\, and\, \lambda_\eta$ are additional unknowns which make the system totally quadratic.

For 3-degrees of freedom we have 3 parameters of motion, one for plunging and two for pitching, and 2 Lagrange multipliers as unknowns. Hence, the iterative equations with variable coefficient matrix and the unknown vector read as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & -x_1 & -b_{1j}\,x_j \\ a_{21} & a_{22} & a_{23} & -x_2 & -b_{2j}\,x_j \\ a_{31} & a_{32} & a_{23} & -x_3 & -b_{3j}\,x_j \\ 2x_1 & 2x_2 & 2x_2 & 0 & 0 \\ b_{1j}\,x_j & b_{2j}\,x_j & b_{3j}\,x_j & 0 & 0 \end{bmatrix}\begin{pmatrix}\delta x_1\\ \delta x_2\\ \delta x_3\\ \delta\lambda\\ \delta\lambda_\eta\end{pmatrix}=-\begin{pmatrix}f_1\\ f_2\\ f_3\\ f_4\\ f_5\end{pmatrix}$$

Where,

$$\begin{pmatrix}f_1\\ f_2\\ f_3\\ f_4\\ f_5\end{pmatrix}=\begin{pmatrix}(a_{1j}-\lambda\delta_{1j}-\lambda_\eta b_{1j})x_j\\ (a_{2j}-\lambda\delta_{2j}-\lambda_\eta b_{2j})x_j\\ (a_{3j}-\lambda\delta_{3j}-\lambda_\eta b_{3j})x_j\\ x_j\,x_j-1\\ x_i\,b_{ij}\,x_j\end{pmatrix}$$

Here, known values of x_i, $\lambda\; and\; \lambda_\eta$ from the previous iteration level are used in the coefficient matrix and in the load vector.

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