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A NOVEL NONREFLECTING BOUNDARY CONDITION FOR HYPERBOLIC SYSTEMS

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ABSTRACT

In this paper, we introduce a novel nonreflecting boundary condition, called random nonreflecting boundary condition, for use in hyperbolic systems to avoid superious reflections from the boundary. Using well-known high order numerical schemes for transonic nozzle flow with backward acoustic disturbance, implementation of this random nonreflecting boundary condition yields results in excellent agreement with standard nonreflecting boundary conditions.

The nozzle flow is modeled by the Euler equations with variable nozzle area. In the present work, the acoustic wave will be computed directly by solving the non-linear governing equations, rather than solving the linearized equations. Therefore, the need of accurate and efficient numerical algorithms with high truncation order and high resolution has been increased for CAA. For the present work, seven points DRP scheme and the optimized fourth-order compact schemes are used for the evaluation of spatial derivatives and the classical fourth-order Runge-Kutta scheme is used for the integration in time. In this work, there are two different nonreflecting boundary conditions are used for the inflow and outflow boundary conditions. In the case of standard nonreflecting boundary condition that is developed by Thomson by using characteristics wave variables. Alternative nonreflecting boundary conditions are proposed in the present work.

INTRODUCTION

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It is well-known that numerical solutions for time dependent problems of hyperbolic systems require the use of nonreflecting boundary conditions in the numerical scheme that converges to the specified time dependent boundary condition, as it is the case for wave propagation in fluids. Early investigations [Enquist and Majda, 1977; Hedstrom, 1979] dealt with developing nonreflecting boundary conditions for linear hyperbolic systems, even in multiple dimensions. Generalization to quasi-linear and nonlinear hyperbolic systems has also been carried out [Thompson, 1987; Giles, 1990], even for the direct numerical simulation of compressible viscous flows [Pointsot and Lele, 1992]. Alternatively, an absorbing boundary condition technique was developed by adding a buffer/sponge zone to the nonreflecting boundary in which the numerical solutions are filtered or damped [Hu, 2004]. The perfectly matched layer (PML), introduced by [Berenger, 1994] for computational electromagnetics, is a technique used for developing nonreflecting boundary conditions which absorb outgoing waves at open computational domains. Recently the PML technique has been applied for solving nonlinear Euler and Navier-Stokes equations [Hu, Li and Lin, 2008].

In this investigation, we introduce a novel one-dimensional nonreflecting boundary condition, called random nonreflecting boundary condition, for use in hyperbolic systems to avoid superious reflections from the boundary. Using well-known high order numerical schemes [Tam and Webb,1993; Kim and Lee, 1996] for transonic nozzle flow with backward acoustic disturbance, implementation of this random nonreflecting boundary condition yields results in excellent agreement with the standard nonreflecting boundary condition of Thompson [Thompson, 1987]. Special non-homogeneous inflow

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and outflow boundary conditions are derived to generate incoming or outgoing disturbances. In the case of standard nonreflecting boundary condition that is developed by Thomson by using characteristics wave variables. Alternative nonreflecting boundary conditions are proposed in the present work such as small random perturbations. It is imposed for the specified functions that refer boundary variables such as density, velocity and pressure at the exit plane.

One-Dimensional Random Nonreflecting Boundary Conditions

In this section we introduce a new nonreflecting boundary condition for one-dimensional unsteady flows based on imposing small random perturbations on the specified boundary conditions to avoid spurious reflections in the artificial computational domain. These small perturbations are generated by the random number generator and are implemented in such a way that they alternate in sign in sucessive time steps. The magnitude of these small perturbations is usually chosen depending on the desired accuracy of the numerical scheme. For the implementation of nonreflecting random boundary conditions in computational domain, let the specified boundary conditions for the acoustic pressure, the acoustic density and the acoustic velocity be specified functions given by p′(t), ρ′(t) and u′(t), respectively.

Let the numerically calculated values of the acoustic pressure at time 1be p'^{n}. The specified value $p'(t_n)$, for sufficiently large *n*, may deviate apprecially from the calculated value if accumulated deviatonic waves reflect from the boundary into the computational domain. To avoid this, we first let $0 \leq RN \leq 1$ be a random number we define an intermediate acoustic pressure p^{*n} by

$$p^{*n} = \left[1-(RN)\right] p'(t_n) + (RN)\, p'(t_{n+1}) \tag{1}$$

We also let $0 < \varepsilon < 1$. we can now introduce the nonreflecting random boundary condition for the calculated value p^{n+1} of acoustic pressure at $t_{n+1} = t_n + \Delta t_n$, where Δt_n is the nth time step, by

$$p'^{n+1} = p'(t_{n+1}) + (-1)^n \varepsilon p^{*n} \tag{2}$$

Similar nonreflecting boundary conditions apply to the acoustic density ρ' and the acoustic velocity u' as,

$$\rho^{*n} = \left[1-(RN)\right]\rho'(t_n) + (RN)\left[\rho'(t_{n+1})\right] \tag{3}$$

$$\rho'^{n+1} = \rho'(t_{n+1}) + (-1)^n \varepsilon\rho^{*n} \tag{4}$$

and

$$u^{*n} = \left[1-(RN)\right]u'(t_n) + (RN)\left[u'(t_{n+1})\right] \tag{5}$$

$$u'^{n+1} = u'(t_{n+1}) + (-1)^n \varepsilon u^{*n} \tag{6}$$

The advantage of this nonreflecting boundary condition is that it can be applied to any time-dependent boundary condition at a boundary point without the need for enlargement of the computational domain. In addition, if ϵ is chosen to be sufficiently small in magnitude, faster convergence can be achieved using the novel random nonreflecting boundary condition. This can result in the reduction of real computational time for the numerical solution of hyperbolic systems.

Transonic Nozzle Flow with Acoustic Disturbunce

As an application of the foregoing CAA schemes and novel random nonreflecting boundary condition, we consider the problem of superimposing a very small amplitude acoustic wave at the exit on the steady isentropic flow of a perfect gas in a quasi-one-dimensional convergent-divergent nozzle and determine its propagation.

The governing equations for quasi-one-dimensional nozzle flows can be written as

$$\frac{\partial \rho}{\partial t} + \frac{1}{A}\frac{\partial(\rho u A)}{\partial x} = 0 \quad \textbf{(7)}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + \frac{\partial p}{\partial x} = 0 \quad \textbf{(8)}$$

$$A\frac{\partial p}{\partial t} + \frac{\partial\left(puA\right)}{\partial x} + (\gamma - 1)p\frac{\partial(uA)}{\partial x} = 0 \quad \textbf{(9)}$$

where ρ , u and p are, respectively, the density, flow speed and pressure normalized as

$$\rho = \frac{\rho_{\dim}}{\rho_\infty}\ ,\ \ p = \frac{p_{\dim}}{\rho_\infty a_\infty^2}\ ,\ \ u = \frac{u_{\dim}}{a_\infty} \quad \textbf{(10)}$$

with ρ_∞ *and* a_∞ denoting, respectively, the density and the speed of sound in the uniform incoming region of the nozzle, γ denoting the isentropic exponent. In Eq. (7) – (9), *x*, *t* and *A*, respectively, denote the normalized axial coordinate, the normalized time and the normalized area, and are given by

$$x = \frac{x_{\dim}}{L}\ ,\ \ t = \frac{L}{a_\infty},\ \ A = \frac{A_{\dim}}{A_{in}} \quad \textbf{(11)}$$

with L denoting the characteristic length (e.g., the nozzle entrance height) and A_{in} denoting the nozzle entrance area. In the normalization Eqs. (10) and (11), subscript *dim* refers to dimensional variables. The system of Eqs. (7)-(9) should be supplemented by appropriate initial and boundary conditions for given nozzle area *A(x)*. In this work, we use the following nozzle cross section area suggested by NASA [Hardin, Ristorcelli and Tam, 1995].

$$A(x) = \begin{cases} 0.536572 - 0198086 e^{-(\ln 2)\left(\frac{x}{0.6}\right)^2}, x > 0 \\ 1.0 - 0.661514 e^{(-\ln 2)\left(\frac{x}{0.6}\right)^2}, x < 0 \end{cases} \quad \textbf{(12)}$$

We now consider transonic flow in the nozzle where the Mach number becomes close to unity at the throat. We investigate the problem of determining the effect of upstream propagation of small amplitude acoustic waves in a steady state transonic flow in the nozzle, whose geometric configuration is shown in Figure 1. We can then write the normalized unsteady transonic nozzle flow solution in the form

$$\rho\left(x,t\right) = \bar{\rho}(x) + \rho'\left(x,t\right) \quad \textbf{(13)}$$

$$u\left(x,t\right) = \bar{u}(x) + u'\left(x,t\right) \quad \textbf{(14)}$$

$$p\left(x,t\right) = \bar{p}(x) + p'\left(x,t\right) \quad \textbf{(15)}$$

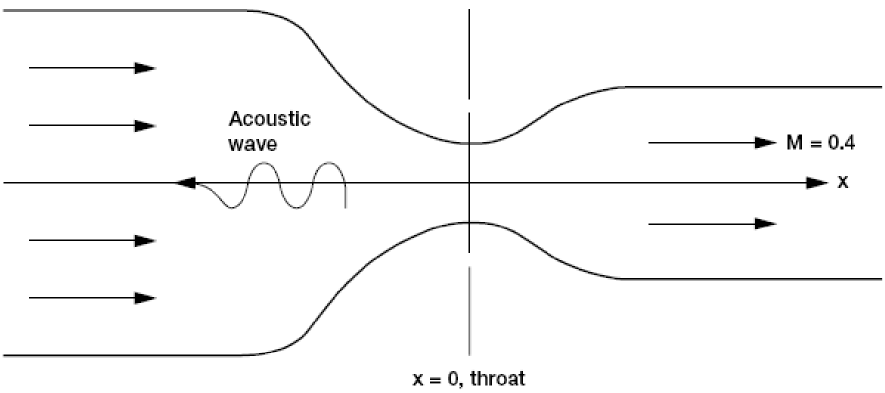

Figure 1: *Propagation of sound through a transonic throat of a transonic nozzle*.

Here, we assume that small amplitude acoustic waves $\left(\rho', u', p'\right)$ are superimposed on the quasi-one-dimensional steady state base field $\left(\bar{\rho}(x), \bar{u}(x), \bar{p}(x)\right)$. We now consider the boundary condition for the acoustic field at the nozzle exit. We let the upstream propagating acoustic disturbance wave at the exit downstream of the nozzle throat be represented by

$$\begin{bmatrix} \rho_e'(t) \\ u_e'(t) \\ p_e'(t) \end{bmatrix} = \varepsilon \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \cos\left[\omega\left(\frac{x_e}{1-M_e}+t\right)\right] \tag{16}$$

where x_e is the location of the nozzle exit, M_e is the nozzle exit Mach number, $\varepsilon = 10^{-5}$ and $\omega = 0.6\pi$. The propagation of the acoustic disturbance wave, expressed by Eq. (16), and its reflection at the nozzle throat can be computed within any desired accuracy by solving the linear system of first order ordinary differential equations subject to the boundary conditions.

One of the most important issues of CAA is the numerical discretization methods. In CAA simulations, a large number of grid points and small time steps are typically required. High-order schemes are thus commonly used for realistic CAA simulations to reduce the number of grid points per wavelength. The numerical schemes are usually originated from traditional schemes and further developed for CAA problems. Among those schemes, Dispersion-Relation-Preserving (DRP) [Tam and Webb,1993] schemes and Optimized Compact (OC) schemes [Kim and Lee, 1996] are discussed in detail and these two schemes are used in the present study.

From a physical point of view, the downstream propagating sound wave will partly be reflected from the area of the transonic nozzle throat and partly transmitted to the upstream of the nozzle throat. In the nozzle throat, the sound wave amplitude will be amplified. Therefore, to ensure that the computed solutions are of high quality on the limited computation domain, the non-reflecting boundary condition has to be imposed on both sides of the computation domain as shown in Figure 1. Accurate boundary condition implementations are important for successful simulations of flows with acoustic perturbations. For nozzle flow, the inflow boundary conditions should accurately specify the inflow conditions and the outflow boundary conditions must allow the outgoing perturbations to pass without introducing non-physical reflections back into the computational domain.

For nozzle problem, at the upstream boundary, the amplitude of the incoming entropy wave i.e. *(u)* and acoustic wave *(u+c)* are must be zero and the amplitude of the outgoing acoustic wave *(u-c)* is computed from interior points. At the exit boundary, the amplitude of the incoming acoustic perturbation *(u-c)* as well as the amplitude of outgoing entropy wave *(u)* and acoustic wave *(u+c)* are computed.

Therefore, for inflow boundary condition of characteristics are definied as given

$$L_1 = \lambda_1 \left(\frac{\partial p}{\partial x} - \rho c \frac{\partial u}{\partial x} \right)$$

$$L_2 = 0 \tag{17}$$

$$L_3 = 0$$

and for outflow boundary conditions, the characterestics are defined as follows

$$L_1 = \frac{\text{-}2\varepsilon\omega}{1\text{-M}_{\text{out}}} \sin\left[\omega \left(\frac{x_{out}}{1 - M_{out}} + t \right) \right]$$
$$L_2 = \lambda_2 \left(c^2 \frac{\partial \rho}{\partial x} - \frac{\partial p}{\partial x} \right) \tag{18}$$
$$L_3 = \lambda_3 \left(\frac{\partial p}{\partial x} + \rho c \frac{\partial u}{\partial x} \right)$$

The local one-dimensional characteristics non reflecting boundary conditions are used in the outlet and inlet as expressed above. The local one-dimensional relations between the characteristic convection terms and the primitive variables are generated from the wave convection equations. The physical boundary conditions are imposed to the characteristic convection terms using the local one dimensional relations. No extrapolations are needed in the implementation of the present boundary conditions. Full nonlinear Euler equations in their entire conservation forms are directly solved at the boundary without linearization or simplifications.

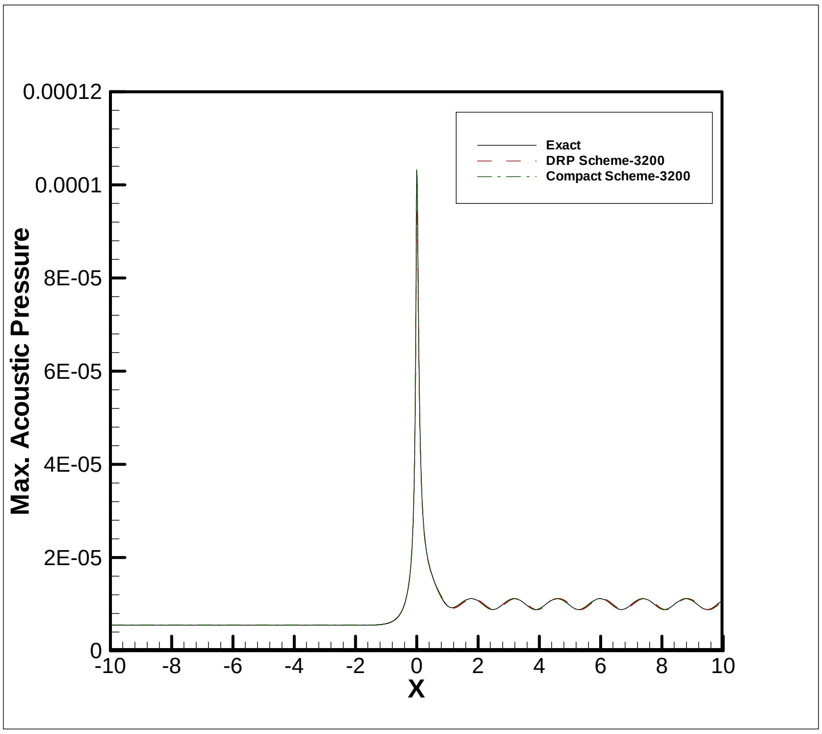

Figure 2: *The maximum acoustic pressure distributions along the nozzle axis for DRP scheme and the optimized compact scheme with uniform grid spacing, N_{max}=3200.*

In addition, the computed maximum acoustic pressures with Dispersion Relation Preserving (DRP) and Optimized Compact (OC) scheme envelopes are then compared with the exact acoustic pressure envelope in Figure 2 for uniform grid and for N_{max} = 3200 grid nodes along nozzle axis. It is obvious that the maximum acoustic pressure distributions are nicely bounded by and touch the exact envelope. Enlarged views of the same figure near the throat and exist are shown in Figure 3. After that maximum acoustic pressure will be used as a comparison criteria.

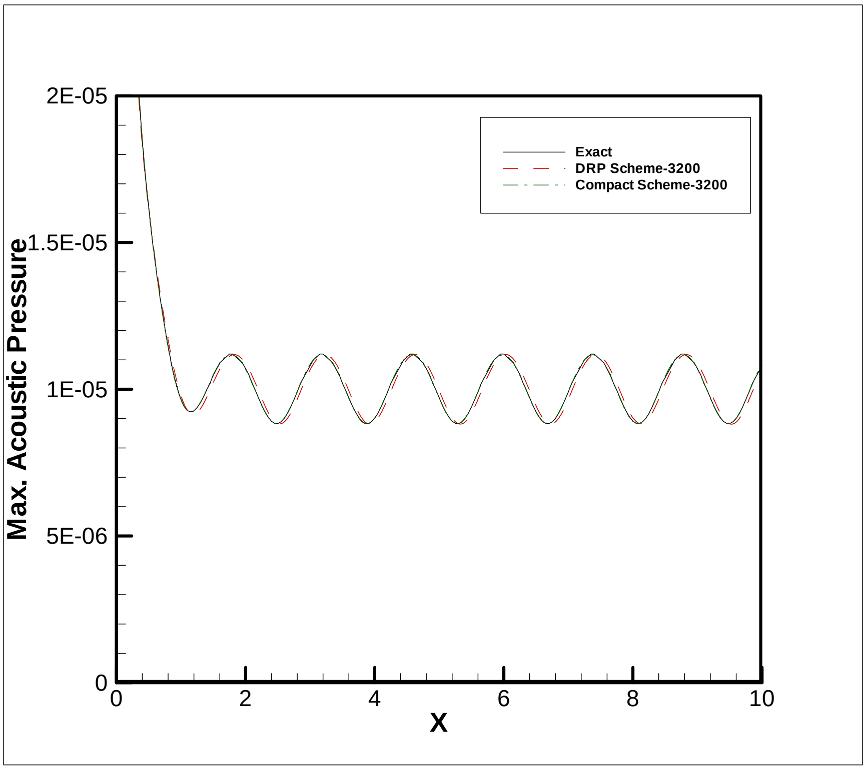

Figure 3: *The maximum acoustic pressure distributions along the nozzle axis for DRP scheme and the optimized compact scheme with uniform grid spacing, N_{max}=3200 at the downstream region (detailed).*

One of the main purposes of problem is to obtain a solution by using as few grid points as possible. Therefore, a non uniform grid technique must be takes into account. This requires the governing equations and the boundary conditions in (x,t) coordinates to be transformed to the new computational domain (ξ,τ) coordinates. In the (ξ,τ) computational domain, the grid spacing is uniform and grid size is taken to be 1 (i.e. Δξ = 1.0). The governing equations are transformed as follows;

The Jacobian of coordinate transformation is defined,

$$J = \frac{1}{\left(\partial x / \partial \zeta\right)} \qquad \textbf{(19)}$$

And $\dfrac{\partial \tau}{\partial t} = 1$, for transformed governing equations. The maximum acoustic pressure perturbation throughout the entire computational domain is plotted in Figure 4 for different non-uniform grid space. In Figure 5, detailed region of computational domain is plotted. Non-uniform computational grid generated on a straight line using the Eriksson stretching function [Eriksson, 1982]. It is written as follows,

$$x = \left(e^{\beta\xi} - 1\right) / \left(e^{\beta} - 1\right), \qquad 0 \le \xi \le 1 \qquad \textbf{(20)}$$

where the controls the degree of clustering near x = 0 with parameter β .

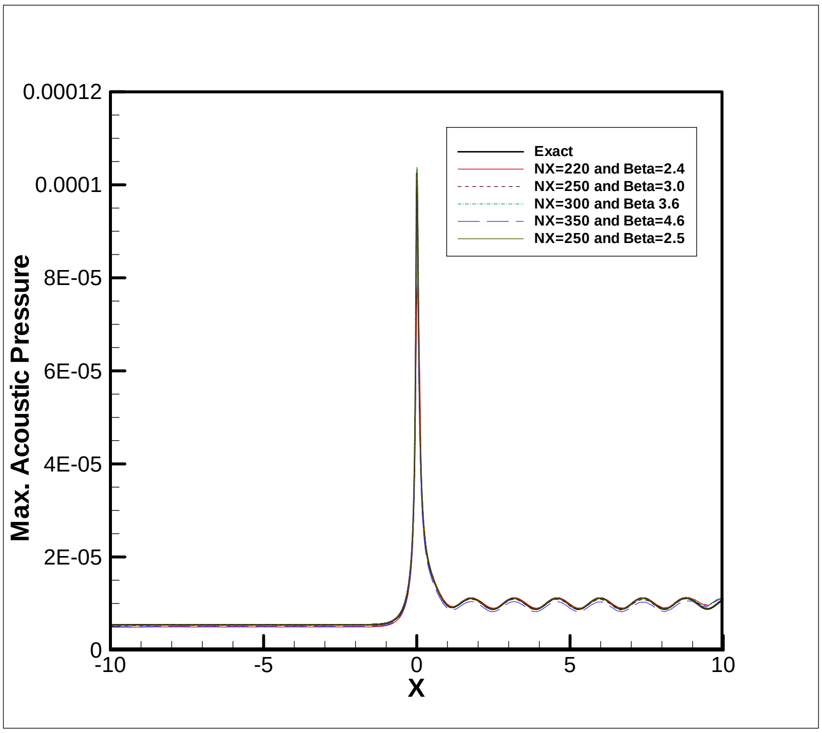

Figure 4: *The maximum acoustic pressure distributions along the nozzle axis for DRP scheme with nonuniform grid spacing for different node number and* β *.*

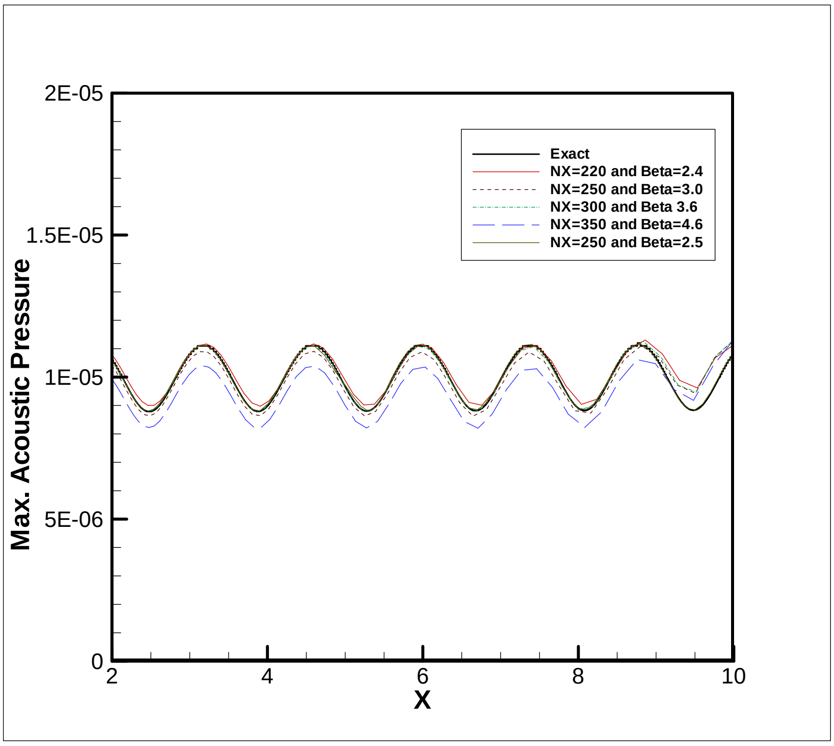

Figure 5: The maximum acoustic pressure distributions along the nozzle axis for DRP scheme with nonuniform grid spacing for different node number and β *at the downstream region (detailed).*

The well agreement is obtained with exact solution for 250 grid points with β = 2.5. Second aim is that random nonreflecting boundary conditions are used in the outlet as expressed above. It is implemented in the last one point for Dispersion Relation Preserving (DRP) scheme and in the last five points for the Optimized Compact (OC) scheme. The maximum acoustic pressure along the entire computational domain and its detailed structure in the nozzle throat and in the exit region is plotted in Figure 6 – 8.

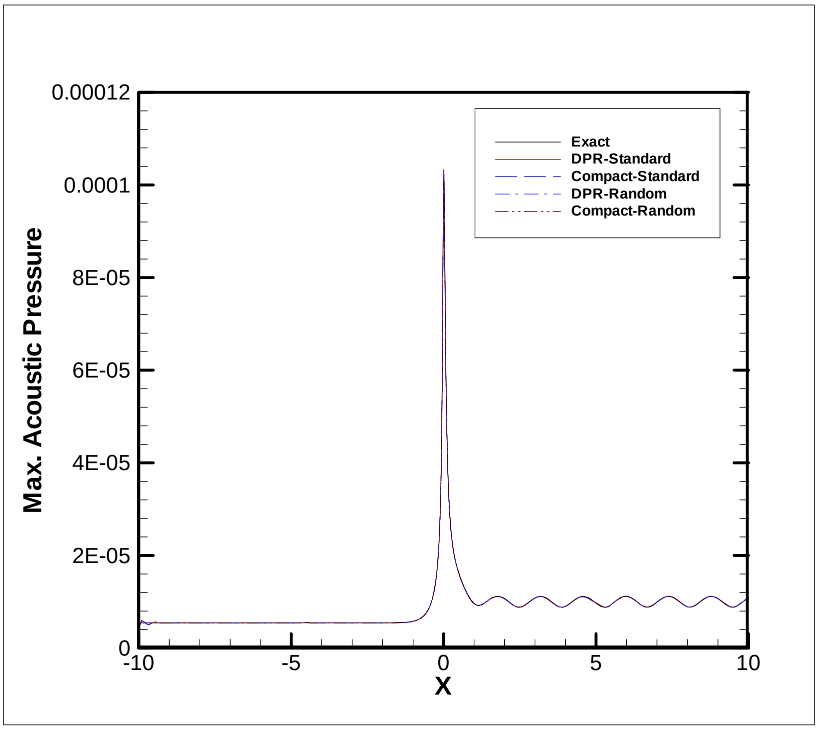

Figure 6: *The maximum acoustic pressure distributions along the nozzle axis for DRP scheme and the optimized compact scheme with nonuniform grid spacing for standard and random nonreflecting boundary conditions*.

For transonic nozzle flow with acoustic disturbance, the inflow boundary conditions should accurately specify the inflow conditions and the outflow boundary conditions must allow the outgoing perturbations to pass without introducing non-physical reflections back into the computational domain. For this reason two different non-reflecting boundary conditions are used. One of them is the well-known standard non-reflecting boundary condition of Thompson, which is based on the solution of the linearized Euler equations by the method of characteristics given in Eq. (17) for the inflow and in Eq. (18) for the outflow. The second one is a novel nonreflecting boundary condition, herein called the random nonreflecting boundary condition, which uses an intermediate value of the acoustic perturbations at the boundary weighted by a random number.

The computed results are obtained by using Ericsonn grid stretching with β = 2.5 and the agreement seems excellent for both schemes using two different nonreflecting boundar conditions, especially demonstrating the validity of the random nonreflecting introduced in this study.

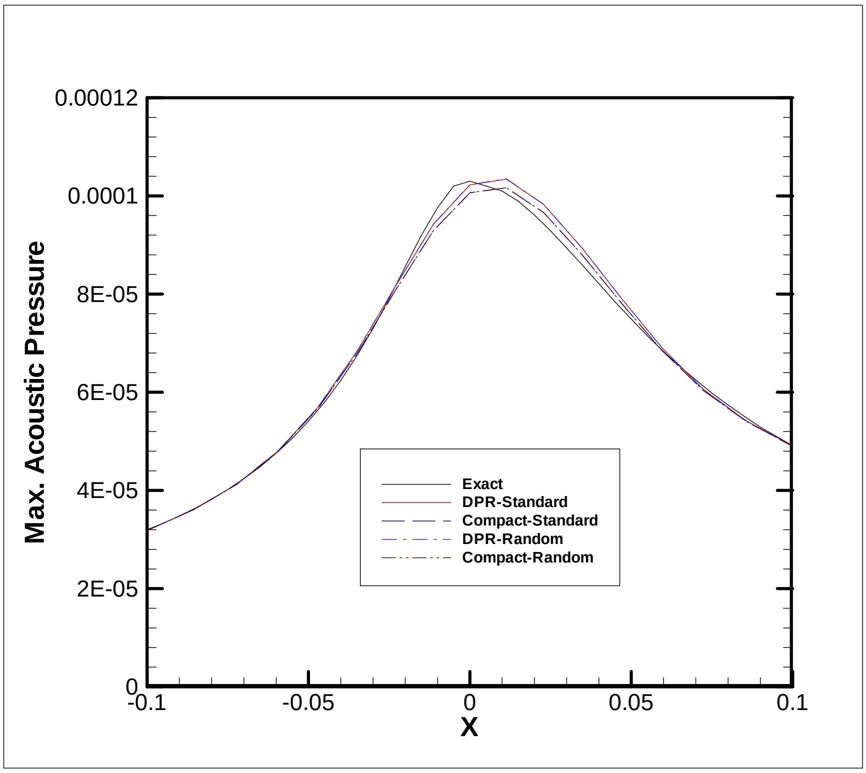

Figure 7: *The maximum acoustic pressure distributions along the nozzle axis for DRP scheme and the optimized compact scheme with nonuniform grid spacing for standard and random nonreflecting boundary conditions at the throat region.*

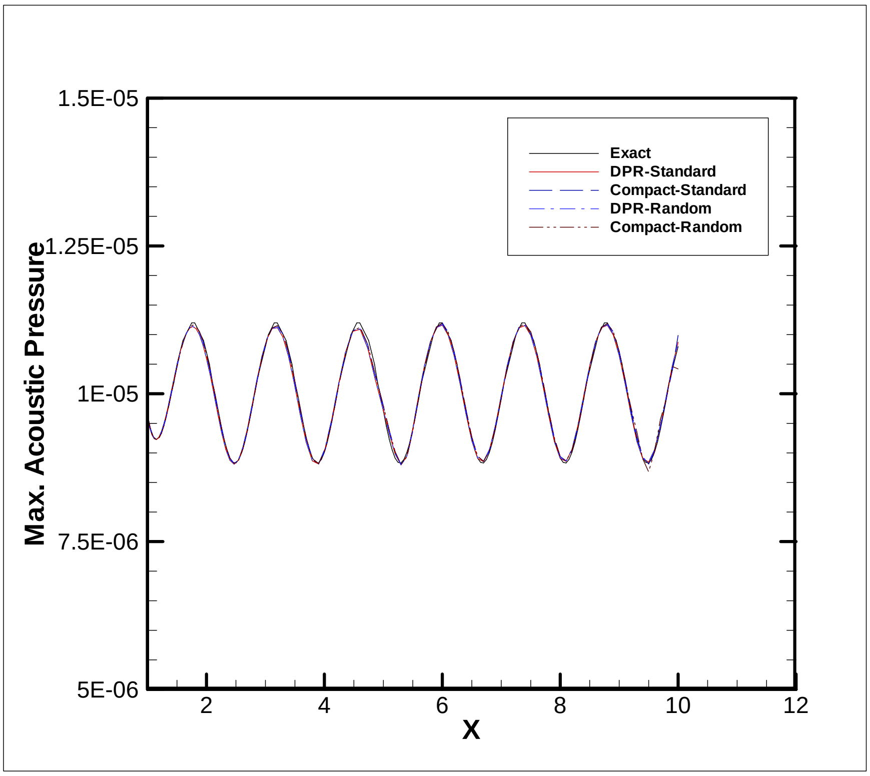

Figure 8: *The maximum acoustic pressure distributions along the nozzle axis for DRP scheme and the optimized compact scheme with nonuniform grid spacing for standard and random nonreflecting boundary conditions at the downstream region.*

CONCLUSION

In this problem the effect of acoustic wave propagation in transonic nozzle flow is investigated by solving the unsteady quasi-one-dimensional transonic nozzle equations in conservative form, where high order computational accuracy is required. For the initial distribution we use the classical quasi-one-dimensional steady-state nozzle flow solution. The acoustic disturbance is implemented at the nozzle exit and is reflected from the nozzle throat. Therefore, for a valid solution, a high order computational scheme with acceptable resolution is required. For this reason we use high order finite difference computational aeroacoustic schemes, namely the Dispersion Relation Preserving (DRP) scheme of Tam and Webb and the Optimised Compact (OC) Scheme of Kim and Lee. In addition, the implemented boundary conditions must be avoid reflecting small acoustic perturbations back into the computational domain. For this reason two different nonreflecting boundary conditions are used. One of them is the wellknown standard nonreflecting boundary condition of Thompson, which is based on the solution of the linearized Euler equations by the method of characteristics. The second one is a novel nonreflecting boundary condition, herein called random nonreflecting boundary condition, which uses an intermediate value of the acoustic perturbations at the boundary weighted by the random number generator. The novel nonreflecting boundary condition is based on imposing small perturbations on the specified boundary condition has the advantage of reducing the size of the computational domain. Both nonreflecting boundary conditions are implemented in the two computational aeroacoustic schemes used (DRP and OC). The numerical results obtained for each scheme are then compared against those obtained by the exact solution of the unsteady quasi-one-dimensional linearized nozzle flow equations. In particular, the exact and computed maximum pressure envelope and time-dependent pressure distributions show very good agreement.

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