ATTITUDE ESTIMATION OF A RIGID BODY FROM INERTIAL/MAGNETIC MEASUREMENTS USING EXPONENTIAL COORDINATES

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ABSTRACT

In this paper a method for estimation of attitude of a rigid body by integration of kinematic equations is presented. Exponential coordinates are used to project kinematic equation of rigid body from highly nonlinear rotation group to three dimensional linear space enabling the possibility of applying simple numerical integration to compute the attitude. Data from accelerometer and magnetic sensors are used as a low frequency estimate of attitude, combined with integration results to control error, enabling long term real time computation. The stability of proposed algorithm is proven theoretically and demonstrated by numerical simulations.

INTRODUCTION

Estimation of orientation of a rigid body is the first essential step towards designing stabilizing controllers in a wide range of applications such as robotics, UAV's and underwater vehicles. Despite the fact that dynamic equations of attitude (which are in fact the equations of kinematic) without any uncertainty are determined, it is not feasible to apply numerical integration methods to extract attitude data from sensor measurements. Presence of all sorts of noise and unwanted data in the output of sensors combined with digitizing and round off noise and errors of discrete computations make it necessary to consider an error control mechanism to guarantee the correctness of estimation and robustness of real-time computations. Assuming certain kinds of statistical characteristics for noise and uncertainties, Kalman Filter based methods like EKF seek to minimize mean square of error to archive this goal. As an alternative, we have methods based on Wahba's problem like the QUEST algorithm that use optimization methods to minimize a cost function and have the advantage of not being restricted to Gaussian noise. A survey of all nonlinear estimation methods can be found in [1]. Also there exists another completely different approach to the problem which does not use optimizations techniques explicitly, but instead tries to manipulate attitude estimation error dynamics using Lyaponov methods to ensure the convergence of error to zero from any initial conditions. This has the advantage that such an observer can suppress any kind of unwanted contamination without taking into account the nature of disturbance or noise (see [2],[3]). Other than contaminations there is another level of complexity associated with attitude estimation and that is the fact that attitude of rigid body evolves on a non-Euclidean space. The space of rotation matrices (3x3 orthogonal matrices with unit determinant or $SO(3)$) as natural representation of attitude, form a Lie group, meaning that it has geometric structure of a 3 dimensional manifold and algebraic structure of a group with matrix product. Consequently to apply numerical integration methods, in addition to kinematic equations one needs to consider a set of algebraic constraints to get the correct results so that the integration results remain a valid rotation matrix. This increases the amount of computations need to be done in real time at each step.

A relatively new approach in numeric integration of differential equations on Lie groups (called Lie group methods) is to use the projection of differential equations to Lie algebra of Lie group which is a

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linear space and this allows us to apply simple forward numerical integrations [4]. The natural map from Lie group to Lie algebra is the exponential map and in case of rotation matrices the matrix exponential of a skew symmetric matrix is an orthogonal matrix with unit determinant which is a rotation matrix. Conversely the inverse of exponential map (matrix logarithm) is well defined and smooth in the neighborhood of identity defining a chart of $SO(3)$, in other words the exponential map is a three-parameter representation of attitude [5].

In this paper we use exponential coordinates to go from highly nonlinear space of rotations to linear space of three dimensional vectors (note that space of skew symmetric matrices is isomorphic to $\mathbb{R}^3$). This transformation allows us to numerically integrate kinematic equations using measurements of angular velocities without considering any algebraic constraints. An estimate of attitude from accelerometer and magnetic sensors, incorporated in equations, in order to guarantee stability and robustness of numerical computations in presence all kinds of approximation errors and contaminations. After a review of mathematical preliminaries and defining proper terminology, we propose the differential equation of estimator and prove its stability of error dynamic. In final section numerical simulations demonstrating robustness of the proposed algorithm are included.

MATHEMATICAL PRELIMINERIES

Considering two ortho-normal frames as inertial and body-fixed frames, the orientation of rigid body can be determined uniquely by a rotation matrix R in a way that if $\boldsymbol{v}$ is a vector in three dimensional space with $\boldsymbol{v}_b$ and $\boldsymbol{v}_e$ being its representations in body-fixed frame and inertial frame respectively we have:

$$\boldsymbol{v}_e = R\boldsymbol{v}_b \ , \ \ R \in SO(3) \ , \boldsymbol{v}_b, \boldsymbol{v}_e \in \mathbb{R}^3$$

where $SO(3)$ is the space of all 3x3 orthogonal matrices with unit determinant. The three columns of R are three unit vectors of body fixed frame resolved in inertial frame. If rigid body is in continuous rotational motion in space, the rotation matrix becomes a function of time and the following equation expresses the kinematics of motion:

$$\dot{R}(t) = R(t)\boldsymbol{\Omega}_\times \qquad (1)$$

In this equation $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}_\times$ are the angular velocity of rigid body in its vector and skew-symmetric matrix forms:

$$\boldsymbol{\Omega} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \leftrightarrow \boldsymbol{\Omega}_\times = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$$

We have Euler's theorem, which states that every orientation of rigid body can be interpreted as a rotation around an eigenaxis collinear with unit vector $\boldsymbol{n}$ and an angle θ (eigenangle), with respect to inertial frame. We define the vector $\boldsymbol{r}$ to be collinear with $\boldsymbol{n}$ and length θ. The following relations hold [4]:

$$\|\boldsymbol{r}\| = \theta = cos^{-1}\frac{1}{2}(tr(R) - 1) \qquad (2)$$

$$\boldsymbol{n} = \frac{1}{2sin\|\boldsymbol{r}\|}\begin{bmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{bmatrix} \qquad (3)$$

Note that 'tr' is the trace operator and R_{ij} is the element of rotation matrix R. The matrix exponential defined by infinite series allows us to make a link between the rotation matrix and vector $\boldsymbol{r}$:

$$R = exp\,(\boldsymbol{r}_\times) \qquad (4)$$
$$R = I + sin\,\|\boldsymbol{r}\|\,\boldsymbol{n}_\times + (1 - cos\|\boldsymbol{r}\|)\boldsymbol{n}_\times^2 \qquad (5)$$

It has been proven that every open neighborhood of $SO(3)$ can be mapped to $\mathbb{R}^3$ using exp function. This defines a coordinate system on $SO(3)$ called exponential coordinates.

The kinematic equation (1) in terms of $\boldsymbol{r}$ takes the following form:

$$\dot{\boldsymbol{r}} = \boldsymbol{\Omega} + \frac{1}{2}\boldsymbol{r} \times \boldsymbol{\Omega} + \left(\frac{1}{\|\boldsymbol{r}\|^2} - \frac{sin\|\boldsymbol{r}\|}{2\|\boldsymbol{r}\|(1 - cos\|\boldsymbol{r}\|)}\right)\boldsymbol{r} \times (\boldsymbol{r} \times \boldsymbol{\Omega}) \qquad (6)$$

A proof of the above relation is presented in the appendix. The right hand side of this equation has a singularity at $\|\boldsymbol{r}\| = 2\pi$ which is expected, because singularity-free representation of $SO(3)$ with three parameters is not possible. It is also proven that there exists at least one $\boldsymbol{r}$ vector (one skew-symmetric matrix $\boldsymbol{r}_\times$) with $\|\boldsymbol{r}\| \le \pi$, for every point on $SO(3)$ (see [8] for proof). It means that, restriction of (6) to $\|\boldsymbol{r}\| \le \pi$ is the non-unique expression of kinematic without singularity.
To compute the R matrix one needs to integrate (1), which is a differential equation on $SO(3)$. This requires numerical integration of the differential equations together with the algebraic equations defining $SO(3)$. Solving the system of DAE's (Differential Algebraic Equations) requires iterative methods which can be demanding in terms of computational power. Alternatively we propose to integrate (6) which is a differential equation on $\mathbb{R}^3$. By writing kinematic equation in the form of (6) we actually turn the original problem which evolves on a nonlinear space to a problem in a linear space. This allows us to extract attitude from sensor data by using a simple numerical integration method with a sufficiently small time step and a mechanism to keep the error under control.

LOW FREQUENCY ESTIMATE OF ATTITUDE

A set of three-axis magnetometers and three-axis accelerometers fixed in rigid body's centre of gravity, can be used to determine orientation of body-fixed frame with respect to inertial frame.

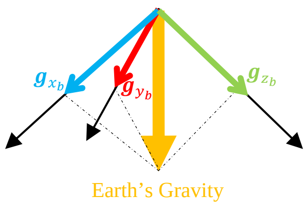

Figure 1: *Gravity Acceleration, Measured by Three Accelerometers Fixed in Body frame*

If we consider the North-East-Down frame as the inertial frame, when there are no linear accelerations the three accelerometers read the vector of earth's gravity resolved in body frame and divided by its norm they provide the 3rd column of rotation matrix R. The same way, the normalized vector of magnetometers readings is nothing but the first column of rotation matrix and finally the 2nd column is the cross product of first and 3rd columns knowing that both frames have the same orientation. There are two problems with this measurement of attitude. First problem arises from the fact that accelerometers actually measure the total linear acceleration. It means that if we are in an accelerated motion other than components of gravity accelerations, linear accelerations are also included in readings and this causes a deviation of inertial frame from its assumed orientation. In addition magnetometers are noisy devices and presence of metallic objects can cause severe distortion in their output. This is a kind of noise that might have non-Gaussian properties making it difficult to remove using Kalman filter and similar techniques.

Consequently accelerometer and magnetometer measurements provide a low frequency estimate of attitude which is not reliable in many cases. On the other hand the angular velocity readings from gyroscopes can be used for real time integration of kinematic equations, but they have their own problems like growing total error caused by digitization and other measurement errors. To solve the problem we use low frequency estimates to compute a correction term added to the kinematic equations to guarantee robustness and stability of numerical computations.

MODIFIED KINEMATIC EQUATIONS

Suppose we have $\hat{\boldsymbol{r}}$ as an estimate of $\boldsymbol{r}$**.** Equation (6) describes dynamic of $\boldsymbol{r}$ resulted from angular velocity. It is reasonable to see the deviation of $\hat{\boldsymbol{r}}$ from $\boldsymbol{r}$ as result of an additive term $\boldsymbol{\Omega}_e(\hat{\boldsymbol{r}},\boldsymbol{r})$ in angular velocity vector $\boldsymbol{\Omega}$. With this assumption we propose the following dynamics for $\hat{\boldsymbol{r}}$:

$$\dot{\hat{\boldsymbol{r}}} = \boldsymbol{\Omega} + \boldsymbol{\Omega}_e + \frac{1}{2}\boldsymbol{r} \times (\boldsymbol{\Omega} + \boldsymbol{\Omega}_e) + \left(\frac{1}{\|\boldsymbol{r}\|^2} - \frac{sin\|\boldsymbol{r}\|}{2\|\boldsymbol{r}\|(1 - cos\|\boldsymbol{r}\|)}\right)\boldsymbol{r} \times \left(\boldsymbol{r} \times (\boldsymbol{\Omega} + \boldsymbol{\Omega}_e)\right) \qquad (7)$$

Note that $\boldsymbol{\Omega}_e$ is a function of $\hat{\boldsymbol{r}}$ which makes the right hand side of (7) a function of $\hat{\boldsymbol{r}}$. Defining error vector $= \hat{\boldsymbol{r}} - \boldsymbol{r}$, we propose the following term for $\boldsymbol{\Omega}_e$:

$$\boldsymbol{\Omega}_e(\hat{\boldsymbol{r}},\boldsymbol{r}) = -k\frac{sin\|\boldsymbol{r}\|}{\|\boldsymbol{r}\|}\boldsymbol{e} \qquad (8)$$

Which $k > 0$ is a design parameter. Subtracting (6) from (7) results in the following error dynamics:

$$\dot{\boldsymbol{e}} = -k\frac{sin\|\boldsymbol{r}\|}{\|\boldsymbol{r}\|}\left(\boldsymbol{e} + \frac{1}{2}\boldsymbol{r} \times \boldsymbol{e} + \left(\frac{1}{\|\boldsymbol{r}\|^2} - \frac{sin\|\boldsymbol{r}\|}{2\|\boldsymbol{r}\|(1 - cos\|\boldsymbol{r}\|)}\right)\boldsymbol{r} \times (\boldsymbol{r} \times \boldsymbol{e})\right) \qquad (9)$$

Now we prove that this error dynamic is always convergent to zero. Inner product of both sides of (9) with $\boldsymbol{e}$ gives:

$$\frac{d\|\boldsymbol{e}\|}{dt} = -k\|\boldsymbol{e}\|\frac{sin\|\boldsymbol{r}\|}{\|\boldsymbol{r}\|}\left(1 - \left(1 - \frac{\|\boldsymbol{r}\|\, sin\|\boldsymbol{r}\|}{2(1 - cos\|\boldsymbol{r}\|)}\right)\left(1 - \frac{|\boldsymbol{e}\cdot\boldsymbol{r}|^2}{\|\boldsymbol{e}\|^2\|\boldsymbol{r}\|^2}\right)\right)$$

The following inequalities hold for $\|\boldsymbol{r}\| \leq \pi$:

$$1 \geq \frac{|\boldsymbol{e}\cdot\boldsymbol{r}|^2}{\|\boldsymbol{e}\|^2\|\boldsymbol{r}\|^2} \geq 0 \quad , \qquad 1 \geq \frac{\|\boldsymbol{r}\|\, sin\|\boldsymbol{r}\|}{2(1 - cos\|\boldsymbol{r}\|)} \geq 0 \quad , \qquad 1 \geq \frac{sin\|\boldsymbol{r}\|}{\|\boldsymbol{r}\|} \geq 0$$

So we have:

$$\frac{d\|\boldsymbol{e}\|}{dt} \leq 0$$

This establishes the fact that the error is non-increasing and exponentially convergent for $\|\boldsymbol{r}\| \neq \pi$. In the other words if we integrate equation (7) instead of (6), in practice the error between estimated value and real value always converges to zero.

As an alternative way of considering the estimation problem, observe that equation (6) is linear in $\boldsymbol{\Omega}$. Now if we consider an additive term like $\boldsymbol{\Omega}_e(\hat{\boldsymbol{r}},\boldsymbol{r})$**,** equation (7) takes the following form:

$$\dot{\hat{\boldsymbol{r}}} = \dot{\boldsymbol{r}} + \boldsymbol{\sigma}(\hat{\boldsymbol{r}},\boldsymbol{r})$$

By choosing a proper term for $\boldsymbol{\sigma}(\hat{\boldsymbol{r}},\boldsymbol{r})$ we can guarantee the stability. For example a $\boldsymbol{\sigma}$ like $\boldsymbol{\Omega}_e$ in (8) gives a convergent error dynamics.

The final question is, how we can know the value of $\boldsymbol{r}$ while that is the value we want to estimate. The answer is the low frequency estimate of $\boldsymbol{r}$ which we indicate with $\boldsymbol{r}_m$ and its associated rotation matrix with R_m. But calculating $\boldsymbol{r}$ might cause problems during numerical computations because we need to normalize a vector with a constantly changing norm which can be zero. To avoid this we use equation (3), so (8) takes the following form:

$$\boldsymbol{\Omega}_e(\hat{\boldsymbol{r}},\boldsymbol{r}) = k\left(\frac{1}{2}\begin{bmatrix} R_{m_{32}} - R_{m_{23}} \\ R_{m_{13}} - R_{m_{31}} \\ R_{m_{21}} - R_{m_{12}} \end{bmatrix} - \frac{sin\|\boldsymbol{r}_{\boldsymbol{m}}\|}{\|\boldsymbol{r}_{\boldsymbol{m}}\|}\hat{\boldsymbol{r}}\right) \qquad (10)$$

But we still need $\boldsymbol{r}$ for computing $\hat{\boldsymbol{r}}$ from (7). This time we can use its estimate, $\hat{\boldsymbol{r}}$. The following diagram is the visual representation of the theory developed in this work.

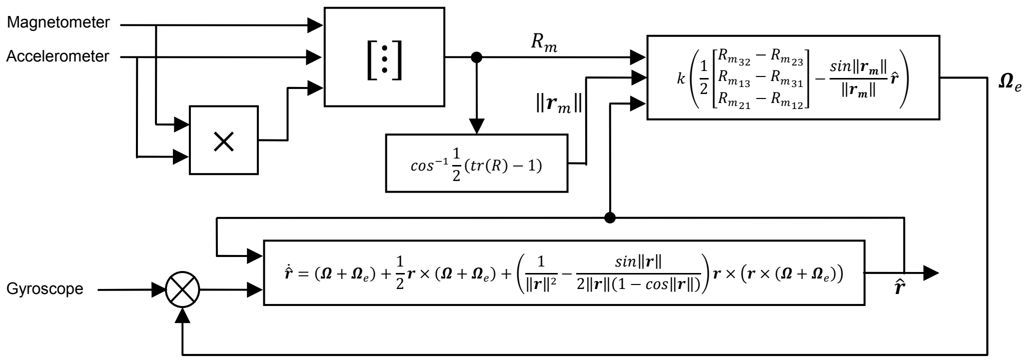

Figure 2: *Block diagram of proposed algorithm for attitude estimation*

NUMERICAL SIMULATIONS

We used SIMULINK for a numerical simulation of the estimator we developed. A model of rigid body exists in SIMULINK standard libraries is used to simulate a rigid body. Outputs of this model are fed to the SIMULINK implementation of our algorithm. We considered a rigid body with J=diag([0.1,0.2,0.3]) and the we set $k = 10$. The following graphs are simulation results of an attitude command of a full yaw turn and then 70 degrees yaw followed roll and pitch commands.

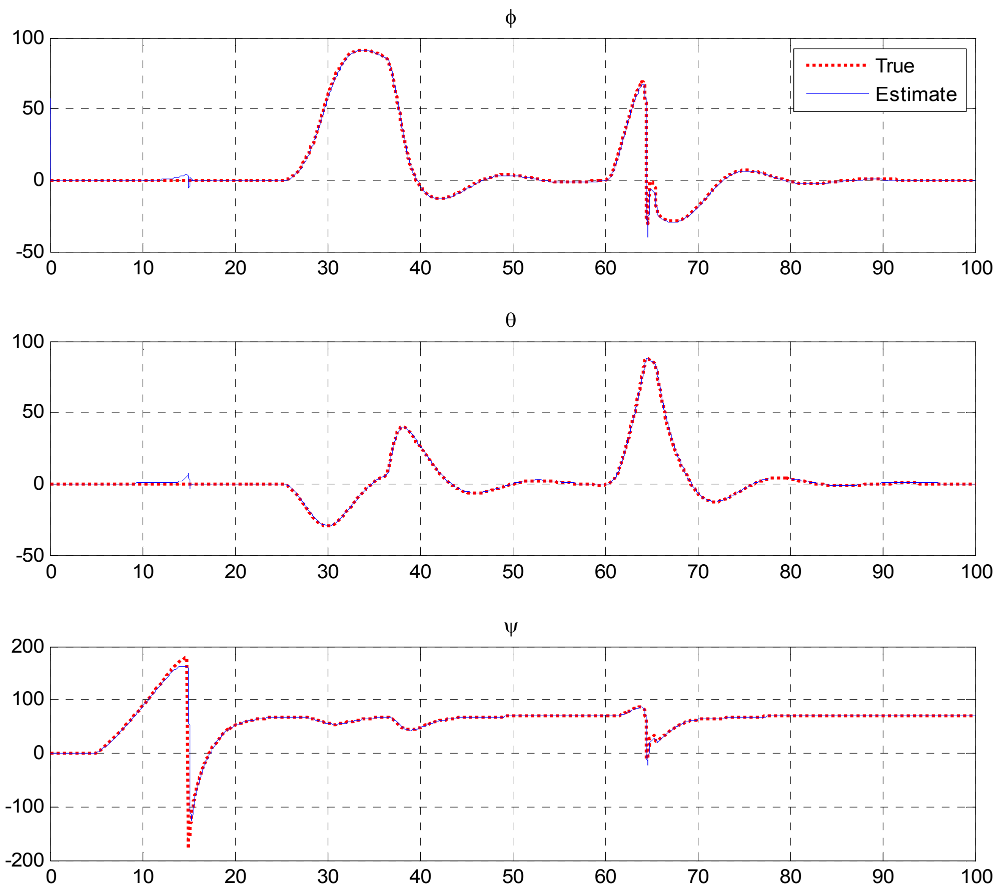

Figure 3: *Tracking of attitude by proposed algorithm indicated by numerical simulation*

CONCLUSIONS AND FUTURE WORK

A method for numerical integration of modified kinematic equations of a rigid body for the purpose of estimation of attitude is presented. Low frequency measurements obtained from three axes accelerometers and magnetometers are used to prevent accumulation of attitude error that would normally occur when gyroscope measurements are directly integrated. Stability of attitude error dynamics is proven. Numerical simulations prove a good tracking of rigid body attitude. In our future research we plan to use a forward integration shceme and optimize the code to save computational power for real time hardware. We also intend to use the method to estimate the attitude of a quadrotor UAV.

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APPENDIX

In this appendix we derive the kinematic equations expressed in exponential coordinates.
We start by differentiating equation (4):

$$R = exp(\boldsymbol{r}_\times) \Rightarrow \dot{R} = \frac{d}{dt} exp(\boldsymbol{r}_\times) = exp(\boldsymbol{r}_\times) \frac{1 - exp\,(ad_{-\boldsymbol{r}_\times})}{ad_{\boldsymbol{r}_\times}} \frac{d\boldsymbol{r}_\times}{dt} = R \frac{1 - exp\,(ad_{-\boldsymbol{r}_\times})}{ad_{\boldsymbol{r}_\times}} \frac{d\boldsymbol{r}_\times}{dt}$$

Where $ad_X Y = XY - YX$ defined for any two square matrices. Using equation (1) we have the following:

$$\frac{1 - exp\,(ad_{-\boldsymbol{r}_\times})}{ad_{\boldsymbol{r}_\times}} \frac{d\boldsymbol{r}_\times}{dt} = \boldsymbol{\Omega}_\times \Rightarrow \frac{d\boldsymbol{r}_\times}{dt} = \frac{ad_{-\boldsymbol{r}_\times}}{exp\big(ad_{-\boldsymbol{r}_\times}\big) - 1} \boldsymbol{\Omega}_\times$$

Just like function $\frac{x}{e^x - 1}$ the right hand side of above can be expanded:

$$\frac{ad_{-\boldsymbol{r}_\times}}{exp\big(ad_{-\boldsymbol{r}_\times}\big) - 1} = \sum_{k=0}^{\infty} \frac{1}{k!} B_k\, ad^k_{-\boldsymbol{r}_\times} = B_0 ad^0_{-\boldsymbol{r}_\times} + B_1 ad^1_{-\boldsymbol{r}_\times} + \sum_{n=1}^{\infty} \frac{1}{(2n)!} B_{2n}\, ad^{2n}_{-\boldsymbol{r}_\times}$$

B_k's are Bernoulli numbers and the only one of them which is not zero for an odd k is $B_1 = -\frac{1}{2}$. Using induction the following can be proven:

$$ad^{2n}_{-\boldsymbol{r}_\times} = (i\|\boldsymbol{r}\|)^{2n-2} ad^2_{-\boldsymbol{r}_\times}$$

so we have the following:

$$\sum_{n=1}^{\infty} \frac{1}{(2n)!} B_{2n}\, ad^{2n}_{-\boldsymbol{r}_\times} = \sum_{n=1}^{\infty} \frac{1}{(2n)!} B_{2n}\, (i\|\boldsymbol{r}\|)^{2n-2} ad^2_{-\boldsymbol{r}_\times} = \frac{-1}{\|\boldsymbol{r}\|^2} \sum_{n=1}^{\infty} \frac{1}{(2n)!} B_{2n}\, (i\|\boldsymbol{r}\|)^{2n} ad^2_{-\boldsymbol{r}_\times}$$
$$= \frac{1}{\|\boldsymbol{r}\|^2}\left(1 - \sum_{n=0}^{\infty} \frac{1}{(2n)!} B_{2n}\, (i\|\boldsymbol{r}\|)^{2n}\right) ad^2_{-\boldsymbol{r}_\times} = \frac{1}{\|\boldsymbol{r}\|^2}\left(1 - Re\left(\sum_{k=0}^{\infty} \frac{1}{k!} B_k\, (i\|\boldsymbol{r}\|)^k\right)\right) ad^2_{-\boldsymbol{r}_\times}$$
$$= \frac{1}{\|\boldsymbol{r}\|^2}\left(1 - Re\left(\frac{i\|\boldsymbol{r}\|}{exp(i\|\boldsymbol{r}\|) - 1}\right)\right) ad^2_{-\boldsymbol{r}_\times} = \left(\frac{1}{\|\boldsymbol{r}\|^2} - \frac{sin\,\|\boldsymbol{r}\|}{2\|\boldsymbol{r}\|(1 - cos\|\boldsymbol{r}\|)}\right) ad^2_{-\boldsymbol{r}_\times}$$

We can put all above together in following form:

$$\frac{d\boldsymbol{r}_\times}{dt} = ad^0_{-\boldsymbol{r}_\times} \boldsymbol{\Omega}_\times - \frac{1}{2} ad^1_{-\boldsymbol{r}_\times} \boldsymbol{\Omega}_\times + \left(\frac{1}{\|\boldsymbol{r}\|^2} - \frac{sin\,\|\boldsymbol{r}\|}{2\|\boldsymbol{r}\|(1 - cos\|\boldsymbol{r}\|)}\right) ad^2_{-\boldsymbol{r}_\times} \boldsymbol{\Omega}_\times$$

This equation is on the skew-symmetric matrices. From the isomorphism of skew-symmetric matrices and three dimensional vectors and equivalence of ad operator for skew-symmetric matrices with cross product for three dimensional vectors have the result:

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{\Omega} + \frac{1}{2} \boldsymbol{r} \times \boldsymbol{\Omega} + \left(\frac{1}{\|\boldsymbol{r}\|^2} - \frac{sin\,\|\boldsymbol{r}\|}{2\|\boldsymbol{r}\|(1 - cos\|\boldsymbol{r}\|)}\right) \boldsymbol{r} \times (\boldsymbol{r} \times \boldsymbol{\Omega})$$