7. ANKARA INTERNATIONAL AEROSPACE CONFERENCE
11-13 September 2013 - METU, Ankara TURKEY

AIAC-2013-143

FINITE-DIFFERENCE IMPLEMENTATION OF LATTICE BOLTZMANN METHOD FOR USE WITH NON-UNIFORM GRIDS

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ABSTRACT

In this paper, finite-difference based Lattice Boltzmann Method (LBM) is implemented to perform fluid flow analysis using non-uniform grids. The implementation is done through the transformation of coordinates from physical space to computational space. With the approach followed, the known limitation of the standard LBM on lattice uniformity is removed. This improvement gives the possibility of solving fluid flow problems with the LBM using coarser grids that are refined where it is necessary. Another advantage gained with the present implementation is handling curved boundaries much easier with non-uniform grids since body-fitted grids might be used now. To validate the present implementation, laminar 2-D lid-driven cavity problem is solved and the results obtained using both uniform and non-uniform grids with the same number of points are compared against each other and with the results from literature. It is observed that clustering the grid points towards to the cavity walls, the solution is improved significantly regarding the velocity profiles along the vertical and horizontal center lines of the square cavity. Then, laminar flow around a 2-D cylinder is solved to show the capability of handling curved boundaries with the current method. For this purpose, some computed quantities are compared with the data from the literature that is obtained experimentally. Very close agreement is obtained for this case, too.

INTRODUCTION

The LBM is a fairly new numerical method which is originated from the Lattice-Gas Automata (LGA) method. The LGA method can be considered as a simple Molecular Dynamics method. The purpose of the LGA method is to simulate the behavior and interaction of particles in a gas as simple as possible. For this purpose, the gas is modeled as a cluster of solid spheres moving along a uniform lattice [Chopard, B. and Droz, M. 2005]. Each solid sphere has a discrete set of possible velocities and the collision between separate particles is handled by a set of elastic collision rules. Macroscopic quantities, such as particle density and velocity at each lattice node, can be computed using the microscopic quantities, making it possible to study the macroscopic behavior of a fluid flow. Even though the idea is simple, the method still gives the same behavior corresponding to the Navier-Stokes equations. Beyond this, it has the advantages such as low memory requirement and high parallelization capability. But numerically, the LGA method suffers the statistical noise caused by the averaging procedure to obtain the macroscopic properties from the microscopic properties.

To remedy the statistical noise that the LGA method suffers, the LBM was developed. As being a derivate of the LGA method, the LBM basically relies on the same idea. But, instead of handling single particles, the LBM handles particle distributions. This removes the need for averaging to obtain the

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macroscopic properties from the microscopic properties, so the statistical noise is also removed. Still, it retains the same advantages as LGA method has [Nourgaliev, R.R. et al 2003]. So, this makes the LBM an attractive method, and there is an increasing interest in the LBM in the Computational Fluid Dynamics (CFD) community. So, the progress in developing and employing LBM is rapid. Even, there are various LBM based commercial flow solvers available on the market, like PowerFLOW developed by Exa Corporation. However, the method is restrictive about the uniformity of the computational grid. This restriction is the property that LBM inherited from the LGA method. These shortcomings are to be the major handicap on widespread use of LBM in engineering problems. So, a lot of research is going on to improve these aspects of LBM.

In this paper, the LBM is implemented using the finite difference approach on generalized coordinates. With the approach followed, the known limitation of the LBM on lattice uniformity is removed and better handling of curved boundaries is succeeded. A generic viscous flow problem, 2-D lid-driven cavity, is solved at low Reynolds number condition and the results obtained using both uniform and non-uniform grids are compared against each other and with the results from literature. Then, laminar flow around a 2-D cylinder is solved to show the capability of handling curved boundaries with the current method.

NUMERICAL METHOD

In the LBM, one solves the kinetic equation of particle distribution function. A kinetic model widely used in the literature is Bhatnagar-Gross-Krook (BGK) model [Bhatnagar, P.L., Gross, E.P., and Krook, M. 1954], which has the following form;

$$\frac{\partial f}{\partial t} + \vec{e}.\vec{\nabla} f = -\frac{1}{\lambda}(f - f^0) \tag{1}$$

where $f = f(\vec{x}, \vec{e}, t)$ is the particle distribution function, in which $\vec{x}$ is the position vector, $\vec{e}$ is the particle velocity vector, t is the time, f^0 is the Maxwell- Boltzmann distribution function, and λ is the relaxation time. To solve f numerically, Equation (1) is discretized in the velocity space using a set of velocities, $\vec{e}_\alpha$ [Bhatnagar, P.L., Gross, E.P., and Krook, M. 1954]

$$\frac{\partial f_\alpha}{\partial t} + \vec{e}_\alpha.\vec{\nabla} f_\alpha = -\frac{1}{\lambda}(f_\alpha - f_\alpha^0) \tag{2}$$

In the above equation, f_α is the distribution function associated with the α^{th} discrete velocity, $\vec{e}_\alpha$. For 2-D problems, one might use D2Q9 model [He, X.Y. and Luo, L.S. 1997b] of which the discrete velocities are shown in Figure (1) below;

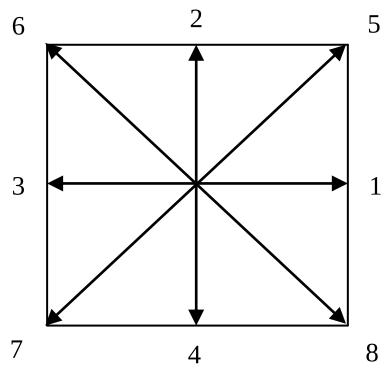

Figure 1. Dicrete velocities of D2Q9 model.

The discrete velocities of D2Q9 model are presented by;

$$
\begin{aligned}
e_0 &= 0 \\
e_{\alpha x,\alpha y} &= c\left(\cos\left((\alpha-1)\pi/2\right), \sin\left((\alpha-1)\pi/2\right)\right) \quad for\ \alpha = 1,2,3,4 \\
e_{\alpha x,\alpha y} &= \sqrt{2}c\left(\cos\left((\alpha-5)\pi/2+\pi/4\right), \sin\left((\alpha-5)\pi/2+\pi/4\right)\right) \quad for\ \alpha = 5,6,7,8
\end{aligned}
\tag{3}
$$

where c is the lattice velocity.

Equation (2) might be written in the following form;

$$\frac{\partial f_\alpha}{\partial t} + \left(e_{\alpha x}\frac{\partial f_\alpha}{\partial x} + e_{\alpha y}\frac{\partial f_\alpha}{\partial y}\right) = -\frac{1}{\lambda}\left(f_\alpha - f_\alpha^0\right) \tag{4}$$

In order to deal with non-uniform grids, a transformation from physical coordinate system (x, y) to generalized coordinate system (ξ, η) is performed. The relation between these two coordinate systems is given by;

$$\begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix} = \frac{1}{J}\begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial x}{\partial \eta} \\ -\frac{\partial y}{\partial \xi} & \frac{\partial x}{\partial \xi} \end{bmatrix} \tag{5}$$

where the Jacobian, J, is defined as;

$$J = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi} \tag{6}$$

Using the chain rule, the convective term in Equation (4) can be written as;

$$\left(e_{\alpha x}\frac{\partial f_\alpha}{\partial x} + e_{\alpha y}\frac{\partial f_\alpha}{\partial y}\right) = \left(e_{\alpha \xi}\frac{\partial f_\alpha}{\partial \xi} + e_{\alpha \eta}\frac{\partial f_\alpha}{\partial \eta}\right) \tag{7}$$

where $e_{\alpha\xi}$ and $e_{\alpha\eta}$ are the contravariant velocities and are defined by;

$$
\begin{aligned}
e_{\alpha\xi} &= e_{\alpha x}\frac{\partial \xi}{\partial x} + e_{\alpha y}\frac{\partial \xi}{\partial y} \\
e_{\alpha\eta} &= e_{\alpha x}\frac{\partial \eta}{\partial x} + e_{\alpha y}\frac{\partial \eta}{\partial y}
\end{aligned}
\tag{8}
$$

So, Equation (4) takes the following form in generalized coordinate system;

$$\frac{\partial f_\alpha}{\partial t} + \left(e_{\alpha \xi}\frac{\partial f_\alpha}{\partial \xi} + e_{\alpha \eta}\frac{\partial f_\alpha}{\partial \eta}\right) = -\frac{1}{\lambda}\left(f_\alpha - f_\alpha^0\right) \tag{9}$$

The macroscopic properties can be calculated using the particle distribution functions using;

$$
\begin{aligned}
\rho &= \textstyle\sum_\alpha f_\alpha \\
\vec{u} &= \frac{1}{\rho}\textstyle\sum_\alpha f_\alpha \vec{e}_\alpha
\end{aligned}
\tag{10}
$$

The convection term in Equation (9) might be discretized using any central or upwind schemes. The central scheme is less dissipative compared to upwind schemes, but it is known to produce unphysical wiggles in the solution field. To remedy this, a mix of central and upwind schemes is used for the present study and 2nd order accuracy is obtained in spatial discretization. The time integration of Equation (9) is done using 2nd order Implicit-Explicit Runge Kutta time discretization scheme [Qian, Y.H., Dhumieres, D., and Lallemand, P. 1992]. The implementation of boundary conditions is done using the extrapolation method giving in [Pareschi, L. and Russo, G. 2005].

VALIDATION OF THE METHOD

First, laminar 2-D lid-driven cavity problem for Reynolds number of 400 is solved both on uniform and non-uniform Cartesian grids of 64x64 points. The obtained solutions are compared against each other and with the results from the literature [Ghia, U., Ghia, K.N., and Shin, C.T. 1982] to show the validity of the current method.

The computed result of Reference [Ghia, U., Ghia, K.N., and Shin, C.T. 1982]in terms of streamlines inside the square cavity at Reynolds number of 400 is shown in Figure (2). The grid used for that computation is a uniform grid with 128x128 points.

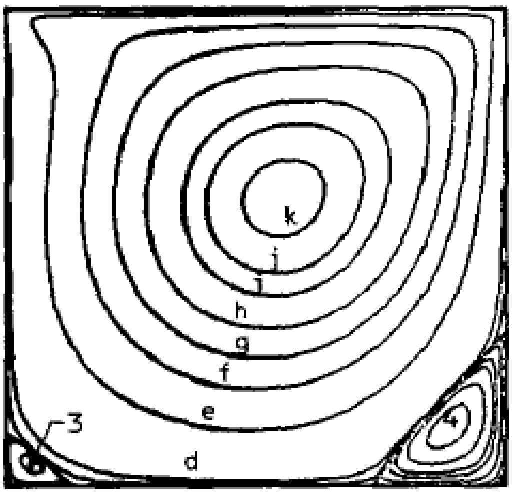

Figure 2. Streamlines inside the cavity for *Re* = 400 on uniform grid of 128x128 points [Ghia, U., Ghia, K.N., and Shin, C.T. 1982]

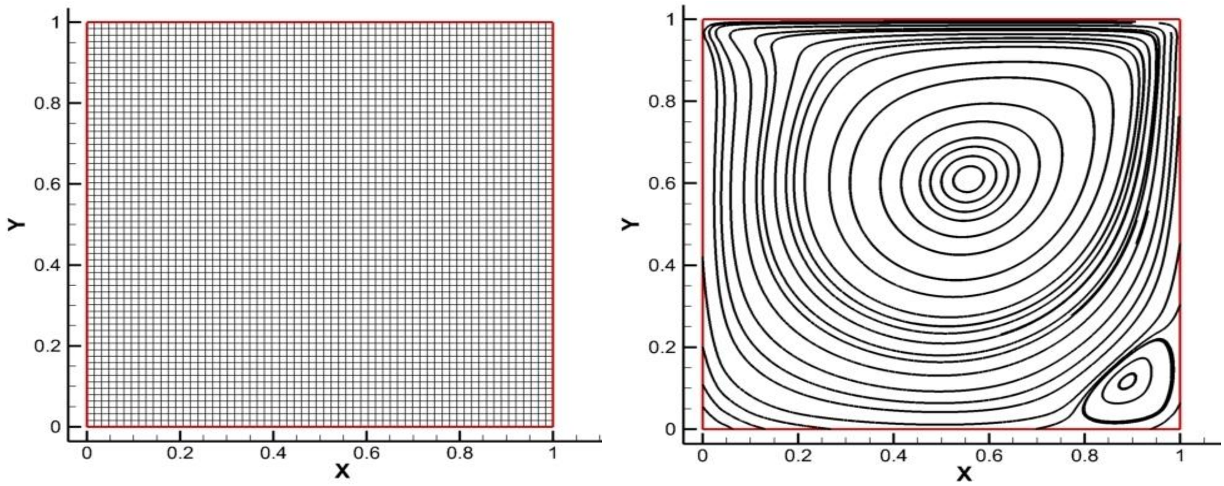

Figure 3. Grid and computed streamlines inside the cavity for *Re* = 400 on uniform grid of 64x64 points.

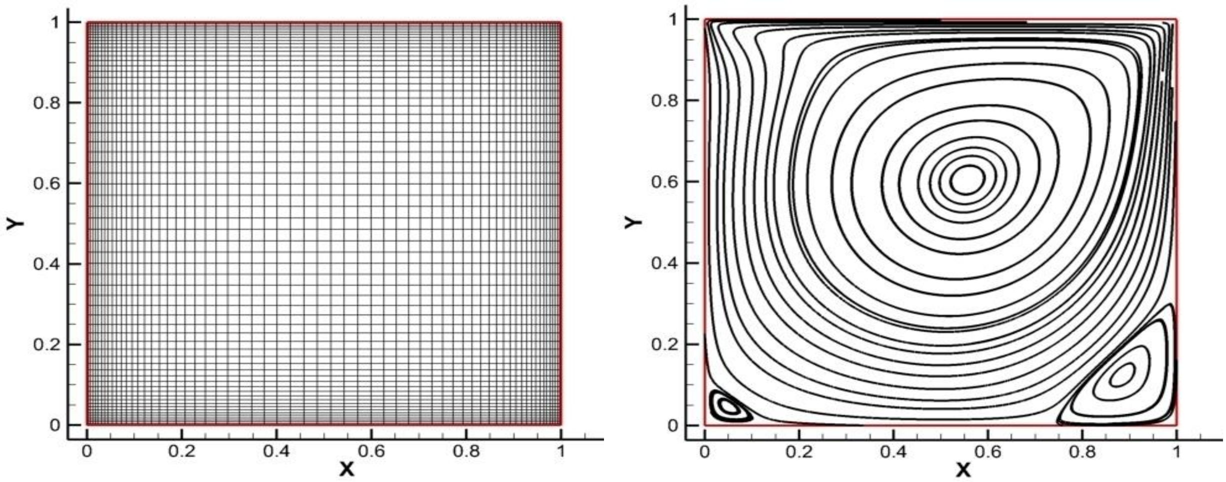

Figure 4. Grid and computed streamlines inside the cavity for *Re* = 400 on non-uniform grid of 64x64 points.

As seen from Figure (2), for this Reynolds number, there exist one primary vortex around the center and two secondary vortices around the bottom corners of the cavity. In the present computations, when uniform grid of 64x64 points is used, the primary and one of the secondary vortices that is at the right corner of the cavity are captured but with smaller magnitude (see Figure (3)). The resolution of the grid is seen to be not enough to capture the left vortex. When a non-uniform grid of 64x64 points is used, the magnitude of the right vortex is increased and the left vortex is captured. This is because of the better resolution around the corners that is obtained with grid stretching.

Comparisons of the velocity profiles along the vertical and horizontal center lines of the square cavity for the grids used and the results of the computations from Reference [Ghia, U., Ghia, K.N., and Shin, C.T. 1982] is shown in Figure (5) and (6), respectively.

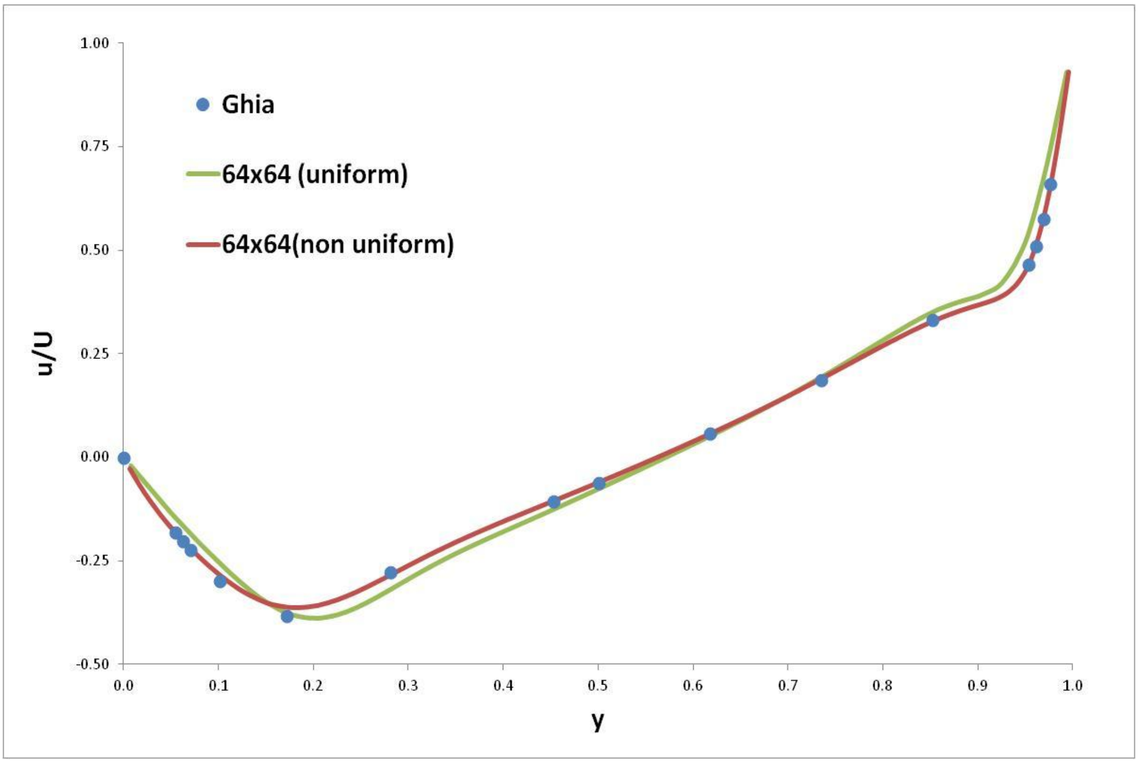

Figure 5. Comparison of velocity component in *x*-direction with the results from [Ghia, U., Ghia, K.N., and Shin, C.T. 1982], for Re=400.

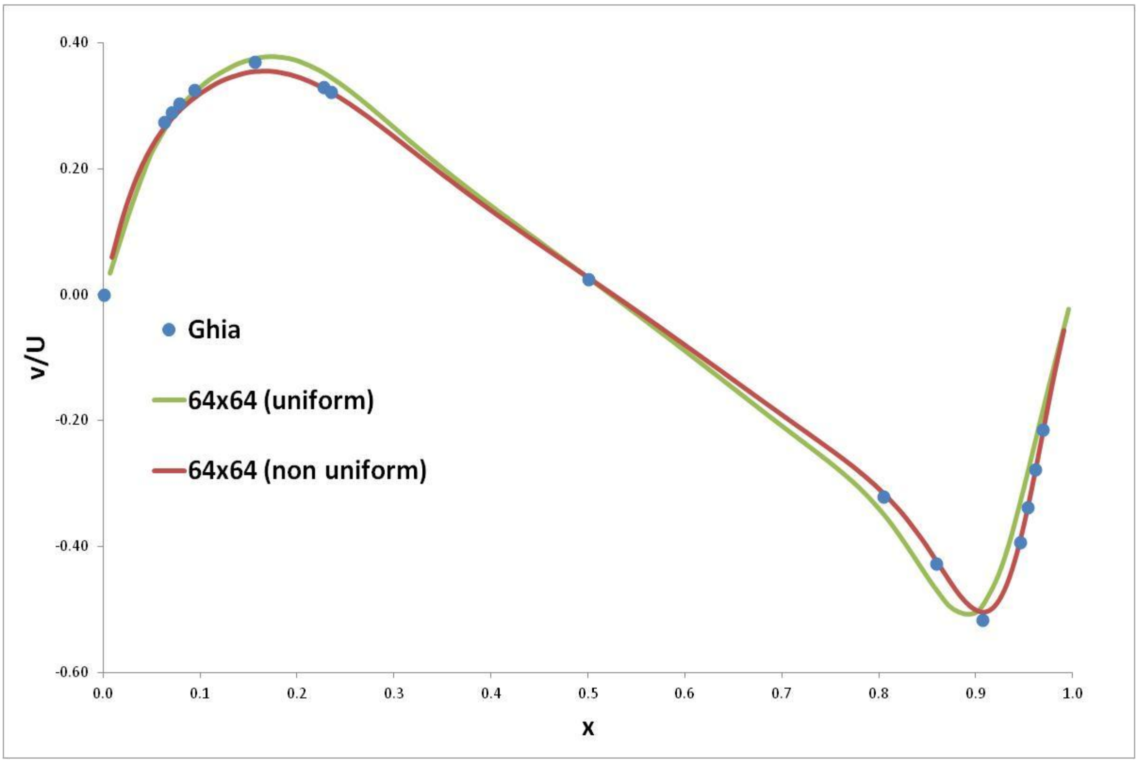

Figure 6. Comparison of velocity component in *y*-direction with the results [Ghia, U., Ghia, K.N., and Shin, C.T. 1982], for Re=400.

As seen from Figure (5) and (6), the velocity profiles are predicted more accurately with non-uniform grid compared to uniform grid that has equal number of points.
To show the capability of the present implementation on handling curved boundaries, the laminar flow around a circular cylinder is solved for various Reynolds numbers ranging from 10 to 40 and the results are compared with the experimental data. The whole computational domain and a closer view of the grid around the cylinder are shown in Figure (7).

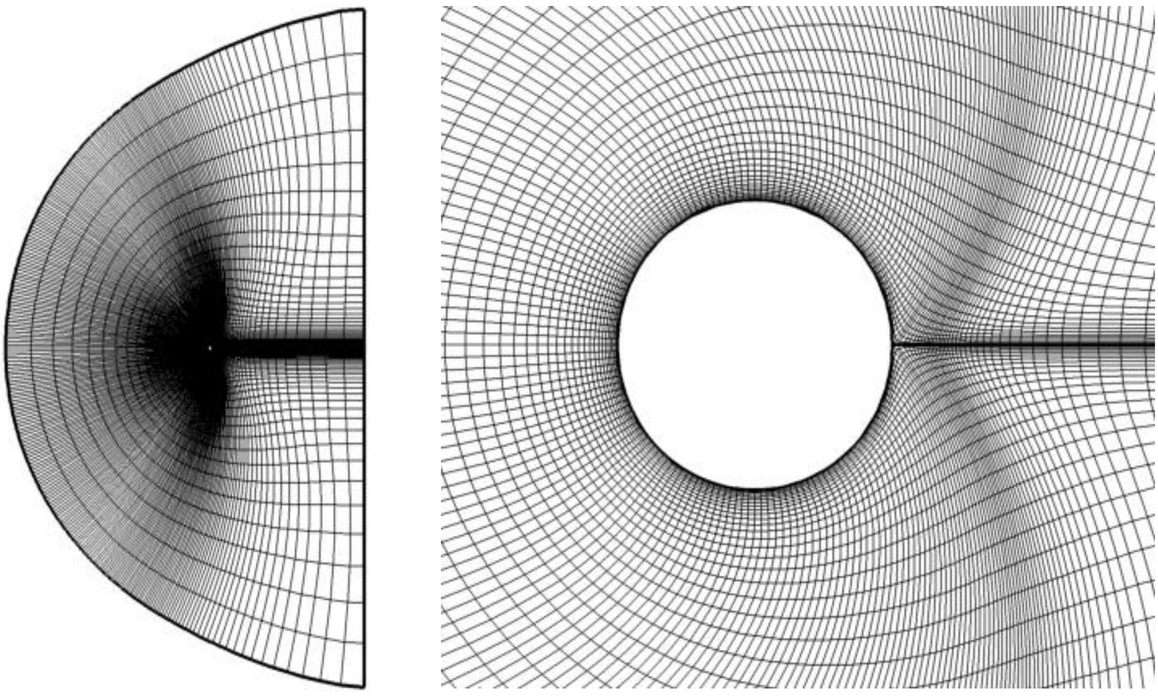

Figure 7. Computational domain and a closer view of the grid around the cylinder.

For the Reynolds numbers up to 40, the flow around the cylinder is steady and the separated flow over the cylinder surface forms two vortices that rotate in opposite direction. The computed streamlines for Reynolds number equal to 10, 20, and 40 are shown in Figure (8).

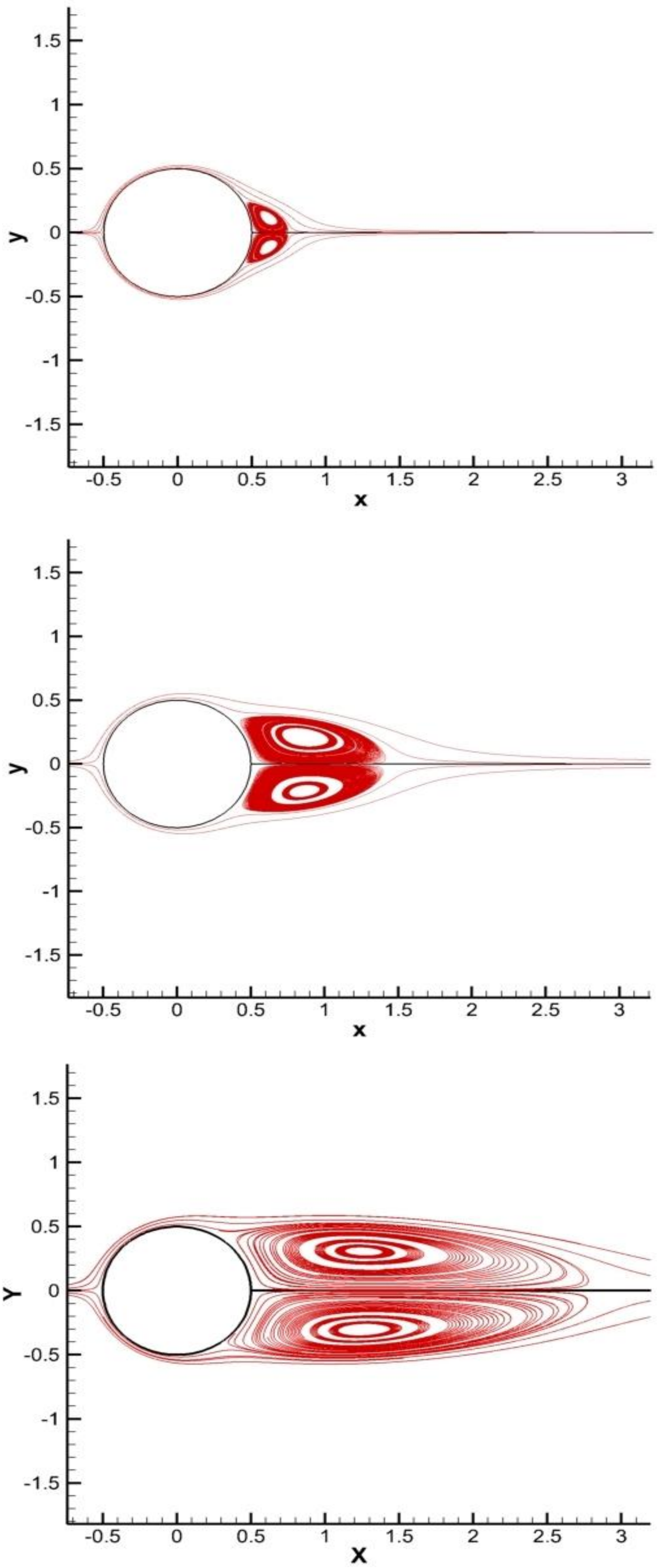

Figure 8. Computed streamlines around the cylinder for *Re* = 10, 20, and 40 respectively.

From Figure (8), the length of the wake and separation angles for the 3 different cases are extracted and compared with the data from the literature as given in Table (1). The computed drag coefficients are also compared as in Table (2). As seen from Table (1) and (2), the present results are in good agreement with the experimental data.

Table 1. Computed and experimental parameters of flow over a cylinder for *Re* = 10, 20, and 40.

	Re=10		Re=20		Re=40	
	L/r	θ	L/r	θ	L/r	θ
[Nieuwstadt, F.T. and Keller, H.B. 1973]	0.434	27.96	1.786	43.37	4.357	53.34
[Coutanceau, M. and Bouard, R. 1977]	0.680	32.50	1.86	44.80	4.260	53.50
[He, X.Y. and Luo, L.S. 1997b]	0.474	26.89	1.842	42.90	4.490	52.84
[Mei, R.W. and Shyy, W. 1998]	0.498	30.0	1.804	42.10	4.380	50.12
Present	0.492	29.11	1.864	43.38	4.357	53.34

Table 2. Computed and experimental drag coefficients of a cylinder for *Re* = 10, 20, and 40.

	Cd		
	Re=10	Re=20	Re=40
[Nieuwstadt, F.T. and Keller, H.B. 1973]	2.828	2.053	1.550
[He, X.Y. and Luo, L.S. 1997a]	3.170	2.152	1.499
Present	2.862	2.051	1.529

CONCLUSION

In this paper, finite-difference based Lattice Boltzmann Method (LBM) is implemented on non-uniform grids. With the approach followed, the known limitation of the standard LBM on lattice uniformity is removed. To validate the present implementation, laminar 2-D lid-driven cavity problem is solved and the results obtained using both uniform and non-uniform grids with the same number of grid points are compared against each other and with the results from literature. It is observed that clustering the grid points towards to the cavity walls, the solution is improved significantly regarding the velocity profiles along the vertical and horizontal center lines of the square cavity. Then, laminar flow around a 2-D cylinder is solved to show the capability of handling curved boundaries with the current method. For this purpose, some computed quantities are compared with the data from the literature that is obtained experimentally. Very close agreement is obtained for this case, too.

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