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AIRCRAFT POSITION AND VELOCITY DETERMINATION BASED ON GPS MEASUREMENTS USING DISTANCE DIFFERENCE AND DOPPLER METHODS

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ABSTRACT

In this study, a comparison is examined between known values of aircraft position, velocity via practicing a motion simulation and calculated values of the same variables via using Distance Difference and Doppler Methods respectively. It is known that the aircrafts are using initial position for navigation on their courses and measuring their velocity instantaneously. Since this information was taken as a reference and absolute values, two methods can be checked for the accuracy according to them. To show the technical background of the methods; Geocentric Coordinates, Global Positioning System and Trilateration Notion is explained at first. As it is mentioned, reference coordinates and velocities of aircraft and satellites are needed and motion simulations of them have been conducted with some assumptions. Next step includes the mathematical theory of the methods, thus Distance Measuring Method (DMM), Distance Difference Method (DDM) and Doppler Method (DM) are explained explicitly. The equations of DDM and DM are hard to solve by hand, therefore Mathematica and Matlab are used to form them. It is essential to indicate how much the methods are accurate so, an error analysis has been conducted to prove the statement. To do so, absolute errors and variances via Delta Method are found. At the end, motion simulations, methods and error analysis are put into a Matlab Code and results are shown graphically. It is understood that the methods are implemented with satisfying accuracy. Additionally, all variance components are checked respect to desired motion parameter to find which one has a greater effect. It turns out that the distance between router satellite and velocity of the satellites are strongly bounded with the aircraft position and velocity respectively. To be sure about this statement, standard deviation values which indicate noise of the measurement has been changed and the components are recalculated. The same results are found with this action. Beside of these examinations, DMM is added to the code and absolute errors between DMM and DDM are found. DMM has to have less accuracy because of the clock bias effect and it is proved via graphically.

INTRODUCTION

As it is mentioned in the study's title, the main purpose of it is determining the aircraft's position and velocity via Distance Difference and Doppler Method and a comparison between reference values and the methods'. It has to be told that these parameters are essential for a single flight and during the flight; these must be measured every time.

Background of Study

Since the very first time of the aviation, the airspeed parameter has been always a big problem as the pioneer aviators noticed that if the aircraft has too low airspeed, it begins to lose the lift force. The speed that the aircraft starts to dive so called stall speed and could be very dangerous if it is in the stall situation.

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Thus, it is considered that the most important flight equipment is pitot static tube that measures the difference between dynamic pressure and static pressure, which relies on Bernoulli Equation.[1] After finding a solution for setting the aircraft in the air, its position issue had been come forward.

First, the navigation system concept must be explained. Navigation System is a measuring system that determines the aircraft position according to either geographical coordinates as they called longitude, latitude and altitude or Descartes coordinates as they indicates “x”, “y” and “z”.[2] For a special case of Descartes coordinates, Geocentric coordinates has been used. A detail statement about this matter is going to be placed in next section.

Navigation Systems have four different types as stated below.[2]

- Geotechnical NS
- Radio technical NS
- Astronomical NS
- Light Based NS

For giving general information, radio technical navigation systems are most common and widely used method of determining the aircraft position nowadays. Hence, to start the basics of this study, it has been going to useful to give brief information about the system.

Radio navigation system is initiated before the WW1 era and the system capabilities highly depended on the technological developments. At that moment, invention of the radio waves and using them for communication was new. Thus, the first thought had been using the transmitted AM radio waves as determining the incoming direction and angle reference. These data are collected via an antenna and plotted on a map so that the pilot could find their intersection as aircraft’s position.[3] However, aircraft’s maximum speed was increasing day by day and even if it was far away from the ground station, the accuracy must have been satisfied. After this RDF concept, a reverser one and a system that can transmit three different phase of signal are developed. The most successful system on bearing-measure called VOR in abbreviation and it has enabled to fly through any willing direction discretely from the others because the rest has been using the low frequency signals.

Furthermore, there are beam navigation systems that aim to keep the aircraft in an airway and the most known version is ILS that leads the aircraft landing on a precise location on the runway. In the limit of all these system, the information is coming out of the air vehicle and the next step was about transmitting the signals also onboard. The RADAR concept is well known on this issue and in use at distance measuring equipment.

The final improvement about radio navigation systems are now off the ground based reference stations and in timeline, it started with military issue LORAN, and then nowadays took part in Global Positioning System as they called hyperbolic measuring. To consider the accuracy of mentioned whole systems, Figure-1.1 is given.[3]

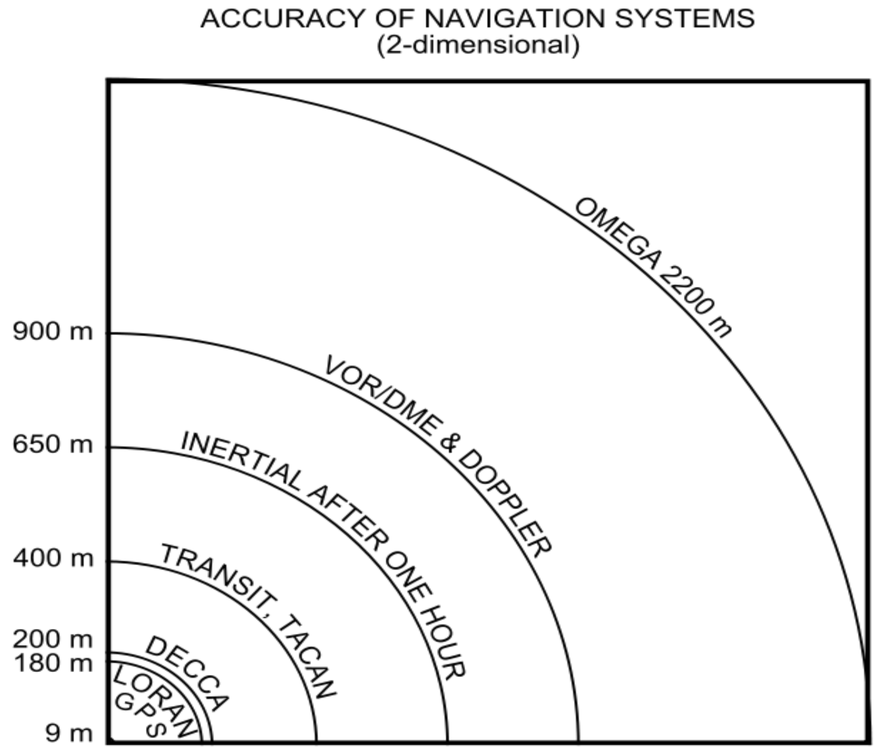

Figure 1.1 : The accuracy of different radionavigation systems.

According to study title, finding aircraft position and velocity is compound of GPS and DME. Hence, there are several advantages that have to be told.

- It has a wide measuring range capability.
- It can be operated in any meteorological situation.
- It measures with high accuracy.
- It can find the aircraft's position and velocity independently from the other systems.[2]

As it can be seen clearly, the last entry tells that with an independency, the measuring systems creates less error because how many equipment they use defines the amount of total error as it forms with gathering the equipment's' own. Therefore, commercial aircrafts have both GPS and a DME; especially an integrated form of distance and angle measuring method. This study has an aim to indicate another solution for the position and velocity.

For better understanding the examining of Distance Difference and Doppler Methods, Geocentric Coordinates, Global Positioning System and Trilateration notions are explained below.

BASIC CONCEPTS

Geocentric Coordinates

Definition of axis systems is varying into several headings but in the scope of aviation literature, there are two basic types. One of them is called geographical coordinates and it indicates the position according to angle from Greenwich meridian that named as longitude, angle from equator that named as latitude and height from the ellipsoidal surface of meridian that named as altitude. This system has a great accuracy about locating an item in the boundary between Earth and space. As a problem of implementation the system, if the item has a great velocity or distance such as satellite or celestial body, measuring the distance is being difficult. On this matter, GPS based navigational systems have been also used geocentric coordinates.

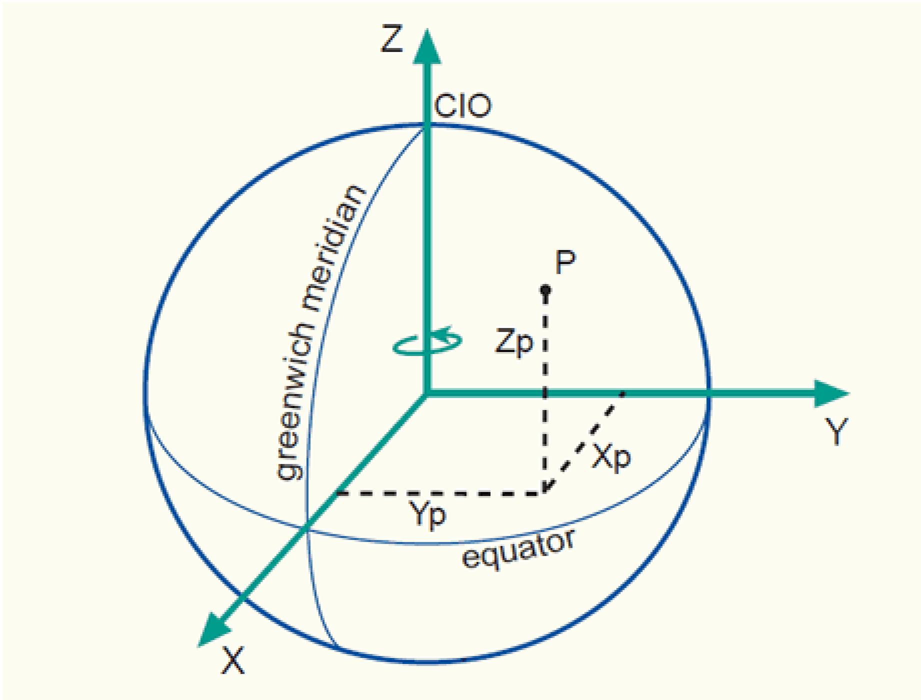

Figure 2.1 : Geocentric coordinate system.

Figure 2.1 shows that the geocentric coordinates formed as X pointed through the intersection of equator and Greenwich meridian, Y is sited on the equator plane and Z is pointed through the north pole.[4] It is important to tell that these are found according to right hand rule. It has to be mentioned that a GPS satellite uses either this type or geographical system and send its location and transmitting time to the receiver for calculating user's location. However, there is a handicap in geocentric coordinates because of the Earth's polar motion as the X and Y-axes are turning around the Z one. The reason of selecting geocentric system is a fact that it decreases the calculation effort and its handicap gives an additional clock bias effect, which it can be eliminated easily via methodically. The clock bias effect is going to be explained later on.

Global Positioning System

GPS system is developed few decades early by USA government because of the navigational requirements as ground based navigation's accuracy highly depends on whether and earth surface conditions.[5] There are 24 satellites in six different orbits which they are equally spaced and inclined 55 degrees. A representative image is given and can be seen on Figure 2.2.[6]

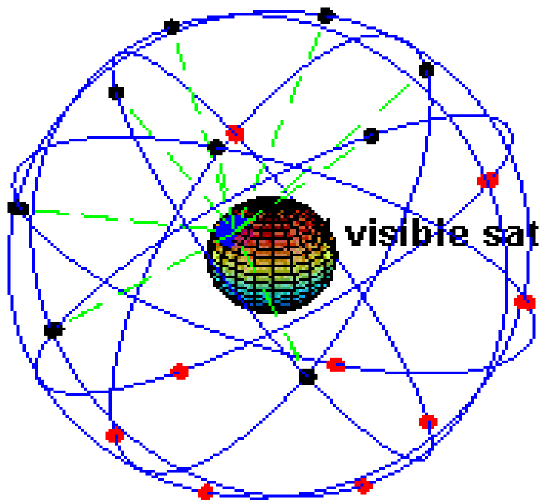

Figure 2.2 : GPS satellites on the Earth's orbit.

As they are positioned uniquely like in Figure 2.2, every user can contact at least four of them every time. That is essential because there have been four variables of position as X,Y,Z and time for a space based navigation.

This system has three subpart called space segment, control segment and user segment. In the space segments, satellites transmit their actual coordinates and the exact time of transmitting action for civil user and military frequency. In user segment, there are receivers to collect this information for calculating the distances that named as pseudorange, however with a time delay called clock bias. This effect as mentioned early occurs as a result of Earth's motion and distance difference between user and the satellite. Time of arrival for the signals measured in nanosecond scale because of the atomic clocks on the satellites but using four satellites multiples the error about computing the location, for instance the error can go up to 300 meters.[5] In control segment, there are correction signals that are sent to satellites for working simultaneously. Hence the signals for receivers have the same amount of error and that kind of issue gives an opportunity to omit it with several ways according to method that is used for measuring the position.

To be clear about the GPS principle, it has to be mentioned that there are three types of using.[7] First, there is an absolute positioning that is used with single pack of code and one receiver. This one is widely in use as civilian. Second one is real time differential positioning that is used with two packs of code and two receivers. The second receiver that its location is well known is broadcasting its own signal and that becomes reference correction information to the actual GPS user. The third one is called real time kinematic positioning and includes the same code and two receivers. Second receiver accepts the data of satellites and then broadcast it with its well-known location information. As it can be seen clearly, if the number entry on the measuring system is increased, the precise and accuracy is also increasing like the amount of errors are minimum 15, 0.5 and 0.01 meters.

For this study, the first version of GPS measuring had been carried on. The logical background of computing the position with four satellites needs another explanation as it is a matter of trilateration concept.

Trilateration

Trilateration is a geometrical issue that provides information about absolute or relative locations via using circles, spheres and triangles.[8] In two-dimensional geometry, if it is known that a point is sited on two curves such as two circles, there are two possible location, which they are intersection of the circles, and center of circles with their radii are enough to find these two locations. Besides, if there is additional information, the point can lie only a unique location. In three-dimensional geometry, if it is known that a point is sited on two surfaces such as two spheres, there are two possible locations, which they are intersection of the spheres, and center of spheres with their radii are enough to find these two locations. If it is known that the point lies on the surface of a fourth sphere then knowledge of this sphere's center with its radius is enough to determine the one unique location. As it can be understood clearly, GPS notion is using three-dimensional trilateration for navigational issues. To represent, Figure 2.3 is given below.[5]

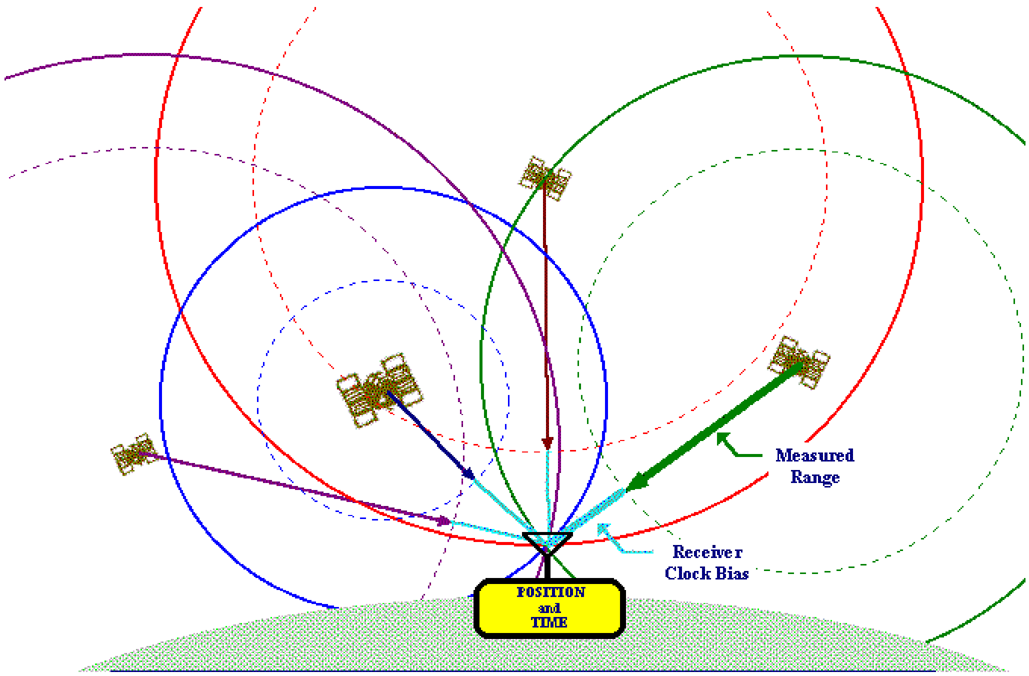

Figure 2.3 : Trilateration with GPS satellites.

In Figure 2.3, four satellites are placed arbitrarily so there are different distances from the user to them. This distance difference creates four spheres that their surfaces and the user is tangent. The reason for necessity of four satellites can be explained like this. If there are three stations on the ground that are going to find the position of aircraft, as it is mentioned early, there will be two possible solutions where the aircraft can be. However, one solution indicates below the Earth's surface so these three stations are sufficient to find one unique solution. For a spaced based navigation, there is a probability of being these two locations so the fourth one will correct the results. In addition to this, there is a clock bias effect because of the GPS method so the position of the aircraft is an action that processing the X, Y, Z and the time difference. In this study, these are examined with easy and accurate methods. The first step is finding the aircraft's reference position and velocity so motion simulation had been executed.

FLIGHT SIMULATIONS

Aircraft Motion Simulation

In the scope of this study, there is a commercial aircraft in a position of level flight. That means its motion occurs only in the X and Y directions. The position and velocity component of the aircraft are remaining stable. Assuming that V is the total speed of the aircraft and it is going to be divided into its component of X and Y. For this matter, the angle of "γ" has to be known and at this juncture, that angle named as sideslip angle. This angle is constant during this level flight. Hence, the velocity component according to geocentric coordinates stated as below.

$$\dot{x} = S\ \cos\gamma \tag{3.1}$$

$$\dot{y} = S\ \sin\gamma \tag{3.2}$$

$$\dot{z} = 0 \tag{3.3}$$

The aircraft is in motion that every unit time, its position is changing and gathering that starts from the reference point. Hence it is stated as in equation **(3.4)**, **(3.5)** and **(3.6).**

$$x_{i+1} = x_i + \dot{x}\,\Delta t \tag{3.4}$$

$$y_{i+1} = y_i + \dot{y}\,\Delta t \tag{3.5}$$

$$z_{i+1} = z_i \tag{3.6}$$

With these equations, the reference values for the aircraft are determined.

Satellite Motion Simulation

GPS satellites that are going to be used in the calculations are in an elliptical orbits and the motion is affected because of the interaction between them and the Earth itself. Hence to point out the relation, Kepler method is going to be used. This system includes three differential form of equation and stated as follow.

$$\frac{d^2x}{dt^2} = -\gamma_K\, M\, \frac{x}{r^3} \tag{3.7}$$

$$\frac{d^2y}{dt^2} = -\gamma_K\, M\, \frac{y}{r^3} \tag{3.8}$$

$$\frac{d^2z}{dt^2} = -\gamma_K\, M\, \frac{z}{r^3} \tag{3.9}$$

Equation **(3.7)**, **(3.8)** and **(3.9)** are indicating the connection of the acceleration and gravitation of the masses. By the way, "γ_K" is the Kepler constant that equals to 6.67×10^{-11} m/kg, "M" is the mass of Earth that equals to 5.976×10^{24} kg, "r" is distance between Earth and satellite center of mass, finally "x","y" and "z" show the geocentric coordinates of satellite. To reveal the equations of velocity and position, these equations must be modified as a differential equation of first degree.

$$\frac{dx}{dt} = U = \frac{x_{i+1} - x_i}{\Delta t}$$

$$\frac{dy}{dt} = V = \frac{y_{i+1} - y_i}{\Delta t} \tag{3.10}$$

$$\frac{dz}{dt} = W == \frac{z_{i+1} - z_i}{\Delta t}$$

$$\frac{dU}{dt} = -\gamma_K M \frac{x}{r^3}$$

$$\frac{dV}{dt} = -\gamma_K M \frac{y}{r^3} \tag{3.11}$$

$$\frac{dW}{dt} = -\gamma_K M \frac{z}{r^3}$$

Therefore, with equation **(3.10)** and **(3.11)**, six equations are revealed for the satellites' motion. In the end, the velocity is calculated as taking the difference of previous one and the changing rate. Also position is calculated as adding the final location and this velocity that multiplied with the unit time. The equations are presented as follow.

$$U_{i+1} = U_i - \Delta t \, \gamma_K M \frac{x_i}{r_i^3} \tag{3.12}$$

$$V_{i+1} = V_i - \Delta t \, \gamma_K M \frac{y_i}{r_i^3} \tag{3.13}$$

$$W_{i+1} = W_i - \Delta t \, \gamma_K M \frac{z_i}{r_i^3} \tag{3.14}$$

$$x_{i+1} = x_i + \Delta t \, U_i \tag{3.15}$$

$$y_{i+1} = y_i + \Delta t \, V_i \tag{3.16}$$

$$z_{i+1} = z_i + \Delta t\, W_i \tag{3.17}$$

Equations from **(3.12)** to **(3.17)** are the principle of determining the reference position and velocity of the satellites. "r_i" is the variable that same as previous one and can be found according to following equation.

$$r_i^2 = x_i^2 + y_i^2 + z_i^2 \tag{3.18}$$

FINDING AIRCRAFT MOTION PARAMETERS WITH RADIO NAVIGATION

Distance Measuring Method

Distance Measuring Method is relying on the distances between aircraft and three stations.[9] Hence, it has been told before, there are two intersection points coming from three spherical surface as a result of trilateration. To explain the principle, a representative schematic is given as Figure 4.1.

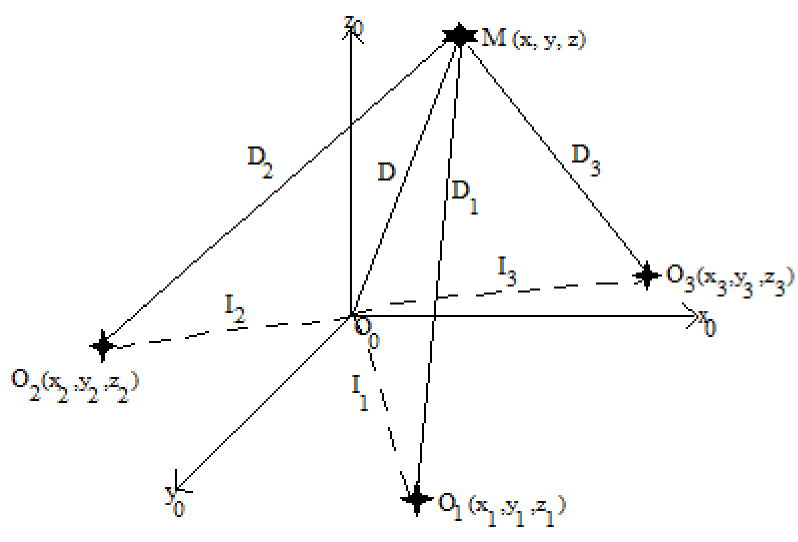

Figure 4.1 : Schematic of distance measuring method.

To be clear, this method is indefinite, because of two intersection points. However, if the stations are on the ground, one point shows beneath of the Earth and creating definite solution, else if the stations are satellites, there must be additional navigational parameter such as one satellite more. Either definite or indefinite, beginning of the method will be same.

As it can be seen on Figure 4.1, the aircraft and ground stations are located on point M, O_1, O_2 and O_3 respectively. In addition, their geocentric coordinates are indicated with the same analogical principle. The distances between stations and the aircraft are represented as D_1, D_2 and D_3; also, aircraft has a distance of D from the origin. Finally, I_1, I_2 and I_3 show the distances between the stations and origin. The first step is writing down the relation of these parameters according to general linear algebra logic.

$$D = \sqrt{x^2 + y^2 + z^2} \quad \textbf{(4.1)}$$

$$\begin{aligned} D_1 &= \sqrt{(x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2} \\ D_2 &= \sqrt{(x_2 - x)^2 + (y_2 - y)^2 + (z_2 - z)^2} \\ D_3 &= \sqrt{(x_3 - x)^2 + (y_3 - y)^2 + (z_3 - z)^2} \end{aligned} \quad \textbf{(4.2)}$$

$$\begin{aligned} I_1 &= \sqrt{x_1^2 + y_1^2 + y_1^2} \\ I_2 &= \sqrt{x_2^2 + y_2^2 + y_2^2} \\ I_3 &= \sqrt{x_3^2 + y_3^2 + y_3^2} \end{aligned} \quad \textbf{(4.3)}$$

The method is continuing with combining equations **(4.1)**, **(4.2)** and **(4.3)** but the satellite based has a different path to follow. As it is mentioned before, GPS navigation uses at least four satellites and has a unique error called clock bias effect. This effect is shown with a symbol of δt.

$$\begin{aligned} (x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2 - (D_1 - \delta t)^2 &= 0 \\ (x_2 - x)^2 + (y_2 - y)^2 + (z_2 - z)^2 - (D_2 - \delta t)^2 &= 0 \\ (x_3 - x)^2 + (y_3 - y)^2 + (z_3 - z)^2 - (D_3 - \delta t)^2 &= 0 \\ (x_4 - x)^2 + (y_4 - y)^2 + (z_4 - z)^2 - (D_4 - \delta t)^2 &= 0 \end{aligned} \quad \textbf{(4.4)}$$

The equation system **(4.4)** is formed for satellite based Distance Measuring Method and one of the ways to solve is Newton-Raphson technique. According to technique, to find the distance in unit time, there must be two forms of matrices that one is equation system **(4.4)** and the other is inverse of the Jacobian matrix which it must be multiplied from left. The next equations present this statement.

$$F = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} \quad \textbf{(4.5)}$$

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} & \frac{\partial f_1}{\partial \delta t} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} & \frac{\partial f_2}{\partial \delta t} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} & \frac{\partial f_3}{\partial \delta t} \\ \frac{\partial f_4}{\partial x} & \frac{\partial f_4}{\partial y} & \frac{\partial f_4}{\partial z} & \frac{\partial f_4}{\partial \delta t} \end{bmatrix} \quad \textbf{(4.6)}$$

Here, “F” indicates the matrix form of equation system **(4.4)** and “J” indicates the Jacobian matrix.

$$\Delta P = -J^{-1}F \quad \textbf{(4.7)}$$

$$P_{K+1} = P_K + \Delta P \quad \textbf{(4.8)}$$

According to equation **(4.7)** and **(4.8)**, “P_{K+1}” and “P_K” show aircraft’s actual position as “ΔP” is the displacement for each interval, however it can be seen easily, this method is an approximation to the real value of result and computing continuously via loop so, there must be a breaking point. To ensure, equation **(4.7)** and **(4.8)** can be applied in a specific amount of interval and the results can be analyzed graphically.

Distance Difference Method

Distance Difference Method is relying on the differences of distances between reference stations and the user.[9] In the scope of this study, stations are selected as GPS satellites, thus the method will be cohered to them. For giving detail information, the trilateral system contains hyperbolic surfaces so they give one unique solution. Discretely from Distance Measuring Method, one of the satellites must be selected as router and the others will be oriented. To do that, the geocentric origin will be offset to the router one. As to be clear about this, the fourth one is chosen.

$$\Delta D_1 = D_4 - D_1$$

$$\Delta D_2 = D_4 - D_2 \quad \textbf{(4.9)}$$

$$\Delta D_3 = D_4 - D_3$$

In equation system **(4.9)**, the symbol “D” is same as previous that shows the distance between the aircraft and satellites. To determine de coordinates of aircraft, equations **(4.1)**, **(4.2)** and **(4.3)** will be combined and they revealed a new one.

$$D_i = \sqrt{I_i^2 + D^2 - 2(x_i x + y_i y + z_i z)} \tag{4.10}$$

Here, the symbol "I" refers the analogy and satellites' number. As the fourth satellite has already been selected for router, equation **(4.10)** becomes like following.

$$\begin{aligned} D_1 - \delta t &= \sqrt{I_1^2 + D_4^2 - 2(x_1 x + y_1 y + z_1 z)} \\ D_2 - \delta t &= \sqrt{I_2^2 + D_4^2 - 2(x_2 x + y_2 y + z_2 z)} \\ D_3 - \delta t &= \sqrt{I_3^2 + D_4^2 - 2(x_3 x + y_3 y + z_3 z)} \end{aligned} \tag{4.11}$$

It has been clear that on the left hand side of the equation **(4.11)**, there is clock bias effect so the next step will be writing the equation **(4.11)** into the equation **(4.9)** and that action omits it from the calculation. After organizing **(4.9)**, the equations below are found.

$$\begin{aligned} x_1 x + y_1 y + z_1 z - D_4 \Delta D_1 &= f_1 \\ x_2 x + y_2 y + z_2 z - D_4 \Delta D_2 &= f_2 \\ x_3 x + y_3 y + z_3 z - D_4 \Delta D_3 &= f_3 \end{aligned} \tag{4.12}$$

Here,

$$\begin{aligned} f_1 &= \frac{1}{2}(I_1^2 - \Delta D_1^2) \\ f_2 &= \frac{1}{2}(I_2^2 - \Delta D_2^2) \\ f_3 &= \frac{1}{2}(I_3^2 - \Delta D_3^2) \end{aligned} \tag{4.13}$$

Equation **(4.12)** is a nonlinear system that contains three different variables and it is very difficult to solve by hand. Hence, Wolfram Mathematica is solved symbolically **(4.12)** and found the unknown variables as stated below.

$$x = \frac{1}{A}\left[\left(f_1 + D_4\,\Delta D_1 \right)\left(y_2\, z_3 - y_3\, z_2 \right) + \left(f_2 + D_4\,\Delta D_2 \right)\left(y_3\, z_1 - y_1\, z_3 \right) + \left(f_3 + D_4\,\Delta D_3 \right)\left(y_1\, z_2 - y_2\, z_1 \right) \right] \tag{4.14}$$

$$y = -\frac{1}{A}\left[\left(f_1 + D_4\,\Delta D_1 \right)\left(x_2\, z_3 - x_3\, z_2 \right) + \left(f_2 + D_4\,\Delta D_2 \right)\left(x_3\, z_1 - x_1\, z_3 \right) + \left(f_3 + D_4\,\Delta D_3 \right)\left(x_1\, z_2 - x_2\, z_1 \right) \right] \tag{4.15}$$

$$z = \frac{1}{A}\left[\left(f_1 + D_4\,\Delta D_1 \right)\left(x_2\, y_3 - x_3\, y_2 \right) + \left(f_2 + D_4\,\Delta D_2 \right)\left(x_3\, y_1 - x_1\, y_3 \right) + \left(f_3 + D_4\,\Delta D_3 \right)\left(x_1\, y_2 - x_2\, y_1 \right) \right] \tag{4.16}$$

Here,

$$A = \left(x_1\, y_2\, z_3 - x_1\, y_3\, z_2 - x_2\, y_1\, z_3 + x_2\, y_3\, z_1 + x_3\, y_1\, z_2 - x_3\, y_2\, z_1 \right) \tag{4.17}$$

Doppler Method

Doppler Frequency Notion

Determining the aircraft position and velocity via Doppler Method is relying on the Doppler Effect.[9] Doppler Effect means the frequency of signals, which they are possessed by a receiver and transmitter, are varying by each other according to their displacement. The absolute value of this frequency variation is called Doppler frequency.

$$F_D = |f_R - f_T| \tag{4.18}$$

Here, “F_D”, “f_R” and “f_T” are Doppler, receiver and transmitter frequencies respectively. This technique is commonly used for finding aircraft’s total velocity by means of radial velocity. To show how it works, the next image is given as representative.[10]

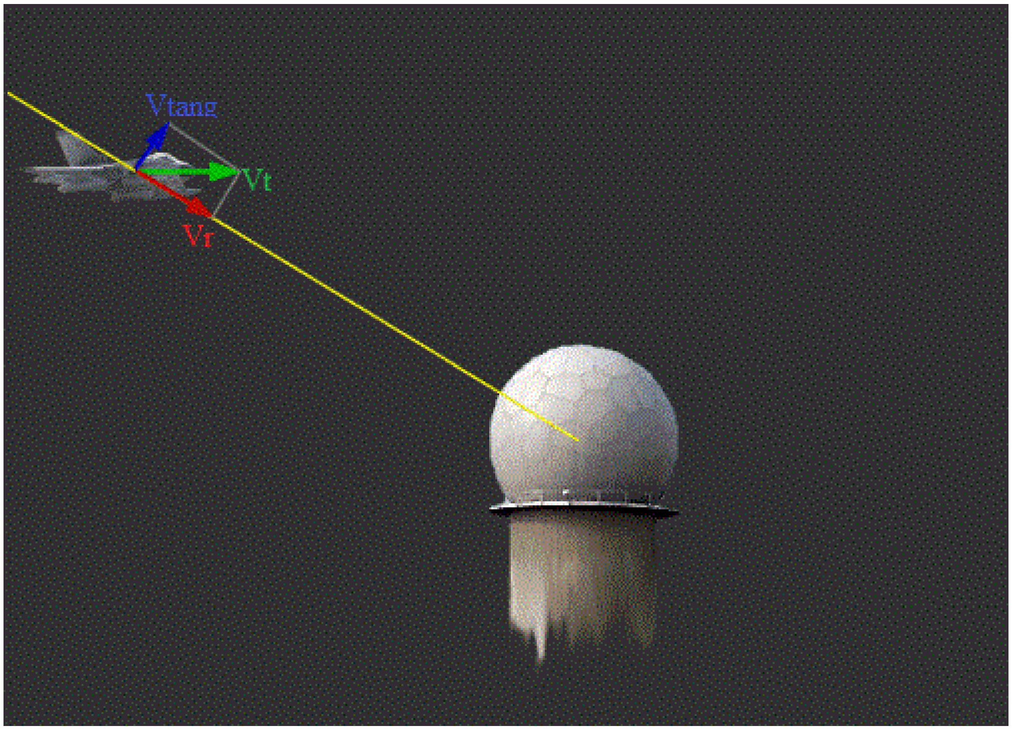

Figure 4.2 : Aircraft velocities according to Doppler Effect.

As it can be seen on Figure 4.2, "V_r" is the radial velocity which is formed either towards the radar or away from it according to aircraft's motion.[10] "V_t" is the total velocity and "V_{tang}" is the tangential velocity. Hence, the radar measures the motion along it and this radial velocity has magnitude variation. To explain this fact, some assumption is going to be made.

Assuming that a constant radar observer acts as a transmitter and signals of it have the frequency of "f_T". If the aircraft, which has a receiver, does not have any displacement according to that location, in a "Δt" time interval the receiver collects an amount of signals that stated as follow.

$$n_0 = f_T \Delta t \tag{4.19}$$

However, the aircraft is in motion and has "V_r" radial velocity so equation (4.19) becomes,

$$n_1 = f_T \Delta t \pm \frac{V_r \Delta t}{\lambda_c} = f_{R_1} \Delta t \tag{4.20}$$

In equation **(4.20)**, "+" sign shows that aircraft is in closing motion, "-"is vice versa. "λ_c" represents the wave length of the transmitting signal and which is equal to radiation speed divided by signal frequency of transmitter. Hence, Doppler frequency and radial velocity can be written as,

$$F_D = \left| f_{R_1} - f_T \right| = \pm \frac{V_r}{\lambda_c} \tag{4.21}$$

$$V_r = F_D \lambda_c = \frac{F_D c}{f_T} \tag{4.22}$$

Here, “c” shows the radiation speed as same as before. If it is considered that the transmitter and receiver located on the same spot, which means aircraft acts as a reflector, the receiver collects signals of the frequency that specified below.

$$f_{R_2} = f_T \left(1 + \frac{V_r}{c}\right)^2 \tag{4.23}$$

In the equation **(4.23)**, there is “V_r^2/c^2” term, which is relatively small amount of value so, it can be omitted. Hence, the receiver frequency becomes,

$$f_{R_2} = f_T \pm 2\frac{V_r}{c} f_T \tag{4.24}$$

Thus, the Doppler frequency and radial velocity are formed as follow.

$$F_D = \left|f_{R_2} - f_T\right| = \pm 2\frac{V_r}{c} f_T = \frac{2V_r}{\lambda_c} \tag{4.25}$$

$$V_r = \frac{c}{2}\frac{F_D}{f_T} \tag{4.26}$$

Determining Aircraft Velocity via Doppler Method

According to Doppler Effect and radial velocity notion, aircraft velocity can be determined via velocity of distance alteration.[9] Hence, for a four satellites system,

$$\begin{aligned}
\dot{r}_1 &= \frac{1}{r_1}\left[(X_1 - x)\left(\dot{X}_1 - \dot{x}\right) + (Y_1 - y)\left(\dot{Y}_1 - \dot{y}\right) + (Z_1 - z)\left(\dot{Z}_1 - \dot{z}\right)\right] + \delta t \\
\dot{r}_2 &= \frac{1}{r_2}\left[(X_2 - x)\left(\dot{X}_2 - \dot{x}\right) + (Y_2 - y)\left(\dot{Y}_2 - \dot{y}\right) + (Z_2 - z)\left(\dot{Z}_2 - \dot{z}\right)\right] + \delta t \\
\dot{r}_3 &= \frac{1}{r_3}\left[(X_3 - x)\left(\dot{X}_3 - \dot{x}\right) + (Y_3 - y)\left(\dot{Y}_3 - \dot{y}\right) + (Z_3 - z)\left(\dot{Z}_3 - \dot{z}\right)\right] + \delta t \\
\dot{r}_4 &= \frac{1}{r_4}\left[(X_4 - x)\left(\dot{X}_4 - \dot{x}\right) + (Y_4 - y)\left(\dot{Y}_4 - \dot{y}\right) + (Z_4 - z)\left(\dot{Z}_4 - \dot{z}\right)\right] + \delta t
\end{aligned} \tag{4.1}$$

Here in equation system **(4.27)**; “$\dot{r}_1$”, “$\dot{r}_2$”, “$\dot{r}_3$”,” $\dot{r}_4$” represent velocity of distance alterations for four satellites individually; X,Y and Z indicate the satellites geocentric coordinates; “$\dot{X}$”(U), “$\dot{Y}$”(V)and “$\dot{Z}$”(W) show the velocities of satellites; "x", "y" and "z" represent the aircraft's coordinates and finally "$\dot{x}$", "$\dot{y}$" and "$\dot{z}$" indicate the aircraft's velocities which they are meant to find. Furthermore,

$$
\begin{aligned}
r_1 &= \sqrt{(X_1 - x)^2 + (Y_1 - y)^2 + (Z_1 - z)^2} \\
r_2 &= \sqrt{(X_2 - x)^2 + (Y_2 - y)^2 + (Z_2 - z)^2} \\
r_3 &= \sqrt{(X_3 - x)^2 + (Y_3 - y)^2 + (Z_3 - z)^2} \\
r_4 &= \sqrt{(X_4 - x)^2 + (Y_4 - y)^2 + (Z_4 - z)^2}
\end{aligned} \qquad \textbf{(4.2)}
$$

In equation **(4.28)**, the symbols are same as before and letter "r" shows the distances between satellites and aircraft. Like in Distance Difference Method, there is clock bias effect in equation system **(4.27)** so difference is taken as the fourth satellite is router and the others are oriented.

$$
\begin{aligned}
\Delta\dot{r}_1 &= \dot{r}_4 - \dot{r}_1 \\
\Delta\dot{r}_2 &= \dot{r}_4 - \dot{r}_2 \\
\Delta\dot{r}_3 &= \dot{r}_4 - \dot{r}_3
\end{aligned} \qquad \textbf{(4.3)}
$$

As it can be seen clearly that solving nonlinear equation system **(4.29)** is difficult by hand, thus it has been carried out by Mathematica program. However, these are very long equations so they have not been given here.

ERROR ANALYSIS

Absolute Error of Aircraft Position and Velocity

The absolute error is defining the closeness of the results according to real values. In this study, absolute error contains the difference between the calculated values via methodically and reference values which they are found with simulation. Hence for the aircraft position,

$$
\begin{aligned}
\Delta x_{abs.err.} &= x_{sim.} - x_{DDM} \\
\Delta y_{abs.err.} &= y_{sim.} - y_{DDM} \\
\Delta z_{abs.err.} &= z_{sim.} - z_{DDM}
\end{aligned} \qquad \textbf{(5.1)}
$$

Here, sub-indexes "DDM" and "sim." indicates Distance Difference Method and simulation respectively. Similarly, the absolute error about aircraft's velocity can be found like,

$$\Delta U_{abs.err.} = u - \dot{x}$$

$$\Delta V_{abs.err.} = v - \dot{y} \qquad \textbf{(5.2)}$$

$$\Delta W_{abs.err.} = w - \dot{z}$$

First term on the right hand side indicates the velocity that found via Doppler Method and the second term shows the velocity component that found by simulation. Absolute error data are going to be presented later as graphically.

Variance of Aircraft Position and Velocity

Variance is a concept that examines the probability distributions of example values as it measures how these values are spreading out from their mean. The importance of the topic is a fact that it gives information about standard deviation and that shows the accuracy of the result. In short way of speaking, if the values are close to their mean, it indicates that the calculation method is precise. There are many ways to do that however, for this study, it has to be practiced approximation of functions. Especially if there are more than one variable in the function, Taylor expansion has to be used. This special case is referred as Delta method.[11]

$$var[f(x)] = (f'(E[x])^2\, var(x) \qquad \textbf{(5.3)}$$

Equation **(5.3)** states that variance of a function is formed with first order derivative of function according to every variable and variance of them. Hence, to begin with the variance of the aircraft's x coordinate, equation **(4.14)** had been taken.

$$var(x) = \left[\frac{\partial x}{\partial X_i}\right]^2 var(X_i) + \left[\frac{\partial x}{\partial Y_i}\right]^2 var(Y_i) + \left[\frac{\partial x}{\partial Z_i}\right]^2 var(Z_i) + \left[\frac{\partial x}{\partial D_4}\right]^2 var(D_4) \qquad \textbf{(5.4)}$$

Here, X,Y and Z express the satellite's coordinates along with their numbering that it is shown with the "i" symbol. The last term of the equation is distance between the fourth satellite and aircraft. It is known that distance differences between satellites and aircraft are not counting as they have been measured already. Similarly, the variance of aircraft's y and z coordinates are found as follow.

$$var(y) = \left[\frac{\partial y}{\partial X_i}\right]^2 var(X_i) + \left[\frac{\partial y}{\partial Y_i}\right]^2 var(Y_i) + \left[\frac{\partial y}{\partial Z_i}\right]^2 var(Z_i) + \left[\frac{\partial y}{\partial D_4}\right]^2 var(D_4) \qquad \textbf{(5.5)}$$

$$var(z) = \left[\frac{\partial z}{\partial X_i}\right]^2 var(X_i) + \left[\frac{\partial z}{\partial Y_i}\right]^2 var(Y_i) + \left[\frac{\partial z}{\partial Z_i}\right]^2 var(Z_i) + \left[\frac{\partial z}{\partial D_4}\right]^2 var(D_4) \qquad \textbf{(5.6)}$$

Besides that, variance of aircraft's velocity can be found with the same logic so, the variances are;

$$var(V_x) = \left[\frac{\partial V_x}{\partial X_i}\right]^2 var(X_i) + \left[\frac{\partial V_x}{\partial Y_i}\right]^2 var(Y_i) + \left[\frac{\partial V_x}{\partial Z_i}\right]^2 var(Z_i) + \left[\frac{\partial V_x}{\partial U_i}\right]^2 var(U_i) + \left[\frac{\partial V_x}{\partial V_i}\right]^2 var(V_i) + \left[\frac{\partial V_x}{\partial W_i}\right]^2 var(W_i) + \left[\frac{\partial V_x}{\partial r_i}\right]^2 var(r_i) + \left[\frac{\partial V_x}{\partial x}\right]^2 var(x) + \left[\frac{\partial V_x}{\partial y}\right]^2 var(y) + \left[\frac{\partial V_x}{\partial z}\right]^2 var(z) \quad \textbf{(5.7)}$$

$$var(V_y) = \left[\frac{\partial V_y}{\partial X_i}\right]^2 var(X_i) + \left[\frac{\partial V_y}{\partial Y_i}\right]^2 var(Y_i) + \left[\frac{\partial V_y}{\partial Z_i}\right]^2 var(Z_i) + \left[\frac{\partial V_y}{\partial U_i}\right]^2 var(U_i) + \left[\frac{\partial V_y}{\partial V_i}\right]^2 var(V_i) + \left[\frac{\partial V_y}{\partial W_i}\right]^2 var(W_i) + \left[\frac{\partial V_y}{\partial r_i}\right]^2 var(r_i) + \left[\frac{\partial V_y}{\partial x}\right]^2 var(x) + \left[\frac{\partial V_y}{\partial y}\right]^2 var(y) + \left[\frac{\partial V_y}{\partial z}\right]^2 var(z) \quad \textbf{(5.8)}$$

$$var(V_z) = \left[\frac{\partial V_z}{\partial X_i}\right]^2 var(X_i) + \left[\frac{\partial V_z}{\partial Y_i}\right]^2 var(Y_i) + \left[\frac{\partial V_z}{\partial Z_i}\right]^2 var(Z_i) + \left[\frac{\partial V_z}{\partial U_i}\right]^2 var(U_i) + \left[\frac{\partial V_z}{\partial V_i}\right]^2 var(V_i) + \left[\frac{\partial V_z}{\partial W_i}\right]^2 var(W_i) + \left[\frac{\partial V_z}{\partial r_i}\right]^2 var(r_i) + \left[\frac{\partial V_z}{\partial x}\right]^2 var(x) + \left[\frac{\partial V_z}{\partial y}\right]^2 var(y) + \left[\frac{\partial V_z}{\partial z}\right]^2 var(z) \quad \textbf{(5.9)}$$

Here, X, Y and Z represent as same as before and U, V and W indicate the satellites' velocity. In addition to these, "r_i" shows the distances between satellites and aircraft, and "x", "y" and "z" are aircraft's coordinates. Hence their variances are found with equations **(5.4)**, **(5.5)** and **(5.6).**

SIMULATION OF METHODS

So far, the methods of determining aircraft position and velocity, and also their absolute error and variances theory have been analyzed. Now an application of theory is going to be done. To do so, some reference values about satellites are needed as it has been told before,thus these are stated in Table 6.1, Table 6.2.

Table 6.1 : Satellites' initial positions.

Satellite Number	X (m)	Y (m)	Z (m)
1	10000000	20000000	23000000
2	22500000	22000000	15000000
3	21000000	18000000	20000000
4	15000000	10000000	18000000

Table 6.2 : Satellites' initial velocities.

Satellite Number	U (m/s)	V (m/s)	W (m/s)
1	2000	2000	2500
2	2000	1500	3027
3	2000	1400	3114
4	2000	1500	3027

In addition, aircraft initial specifications are given in Table 6.3.

Table 6.3 : Aircraft initial specifications.

Aircraft Spec.	Initial Coordinates	Initial Total Velocity	Sideslip Angle
x (m)	0	195 m/s	15˚
y (m)	0		
z (m)	6378000		

For starting the application, equations **(3.1)-(3.6)** and **(3.12)-(3.18)** are used for satellites and aircraft's motion simulation and the reference values have been found. Here, the time interval selected as 1 second.

It is important to say that, the distances between satellites and aircraft can be measured now so, the left hand side of equation **(4.9)** is a known parameter. However, this navigation parameter must have a disturbance because of the signal qualification. Hence, when measuring the variables on the left hand side of equation **(4.2)**, they will have additionally standard deviation of their own multiplied by "randn" values, which is a Matlab command for creating this disturbance mathematically. At the end, equations **(4.14)-(4.17)** are used to find aircraft position via Distance Difference Method but, again it must be told that these equations had to be modified via changing the geocentric reference coordinates to fourth satellite origin coordinates as the method needs. Thus, from every satellite coordinates, the difference is taken with fourth one.

For Doppler Method, the left hand side of the equation system **(4.29)** is a measured parameter by reference values and again they have disturbances. The technique is as same as before like adding disturbances to equation **(4.27)**. Thus, equations **(4.27)-(4.29)** are used to find the aircraft velocity via Doppler method.

To practice the analysis of error, equations in Chapter 5 are used. For absolute error of aircraft position and velocity, equations **(5.1)** and **(5.2)** are calculated and plotted. For finding variances, initially, the first order partial differential equations, have been found symbolically via Mathematica. Then with the standard deviations of each variables equations **(5.4)-(5.9)** are calculated and plotted.

To investigate which group of variables (e.g. satellites' coordinates, satellites' velocities, distances, aircraft position) affect the variance mostly, some another graphics are plotted because the more they affect variances, the more accuracy is relying on.

Finally, to compare distance measuring and Distance Difference Method, the same technique is used as mentioned before via calculation of equations **(4.5)-(4.8)** and absolute error has been found. To ensure, the standard deviations, which they have been used in both calculation of variances and disturbances, are given in Table 6.4.

Table 6.4 : Standard deviations for specified variables.

Standard Deviation	**Satellite Coordinates (X, Y, Z)**	**Satellite Velocities (U, V, W)**	**Sat.-A/C Distance (D)**	**Dist. Diff. Velocity ($r^{\cdot}$)**
	1 m	0.02 m/s	10 m	0.02 m/s

Now the next step will be the presentation of results, which they have been practiced according to previous chapters.

First, Figure 6.1 is indicating the position of aircraft, which they have been found by both simulation and Distance Difference Method. Each line is prepared to show the “x”, “y” and “z” geocentric coordinates respectively.

Secondly, Figure 6.2 is presenting the velocity of aircraft, which again found via simulation and Doppler method. This time, each line is indicating the velocity component along “x”, “y” and “z” direction respectively. These both graphical figures state that methods found the values in desired limit.

To investigate how much the reference and calculated values are close, absolute error graphics are plotted as Figure 6.3. According to this figure, aircraft position components vary around 150, 100 and 200 meters respectively. For an unfiltered measuring, these values are indicating satisfied accuracy. Furthermore, aircraft velocity components vary around 2 m/s so; this value can be also accepted as a good accuracy.

In Figure 6.4, the variances of aircraft position and velocity are given as the accuracy can be also checked from them. For position, the values are varying approximately 50, 60 and 90 meters respectively from the mean and for velocity, they are below 1 m/s. In here, amount of error for “z” coordinate seems higher than the others while this variable is assumed constant as aircraft is in level flight. The reason is estimated that the location of aircraft along the “z” coordinate is continuously further from the center of Earth according to others.

To consider how these results can be better, variance components that are collected in five groups are examined with Figure 6.5-Figure 6.8. These groups contain separately the satellites’ coordinates, velocities, the distance between fourth satellite and aircraft, the distances of aircraft to satellites and the aircraft coordinates. Figures are indicating that both satellites’ coordinates and the router distance are affecting variances of aircraft coordinates. Furthermore, satellites’ velocities are only affecting variances of aircraft velocities.

To be sure about these arguments, standard deviation values had been changed and rechecked. At the end, it has been understood that determining aircraft position is bounded strongly to the distance of router satellite and the aircraft velocity is tied up with the velocities of satellites. If they have been measured in a greater accuracy, aircraft position and velocity can be found more precisely.

Finally, distance-measuring method is practiced to compare the results with Distance Difference Method. To do so, absolute errors are found and tagged as Figure 6.9. In here, around 500 meters error is calculated and it is assumed that this originates from the clock bias effect of the satellites and creates less accuracy.

In the end, the methods are satisfactory with determining the aircraft position and velocity.

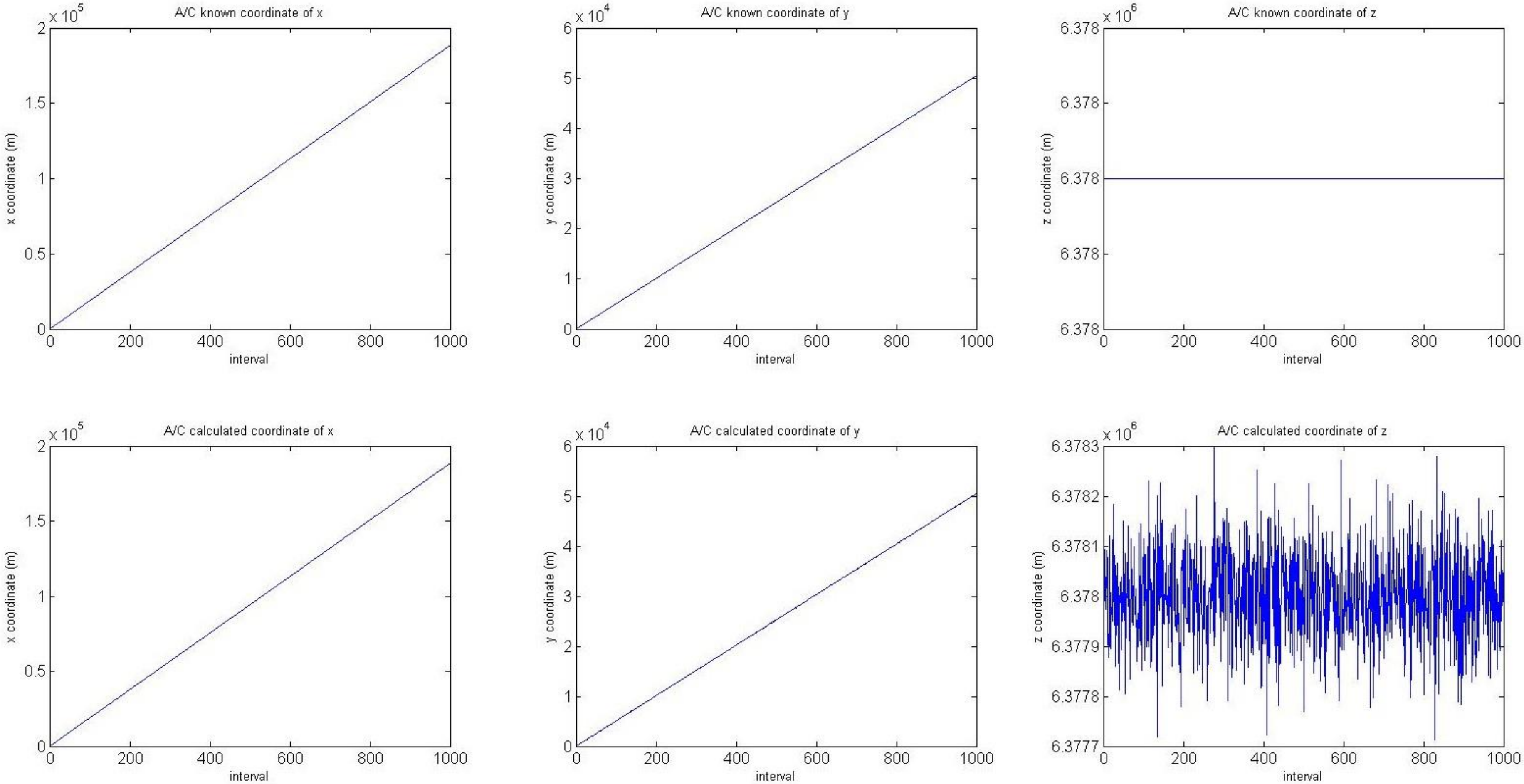

Figure 6.1 : Aircraft reference (simulated) and calculated (via Distance Difference Method) coordinates.

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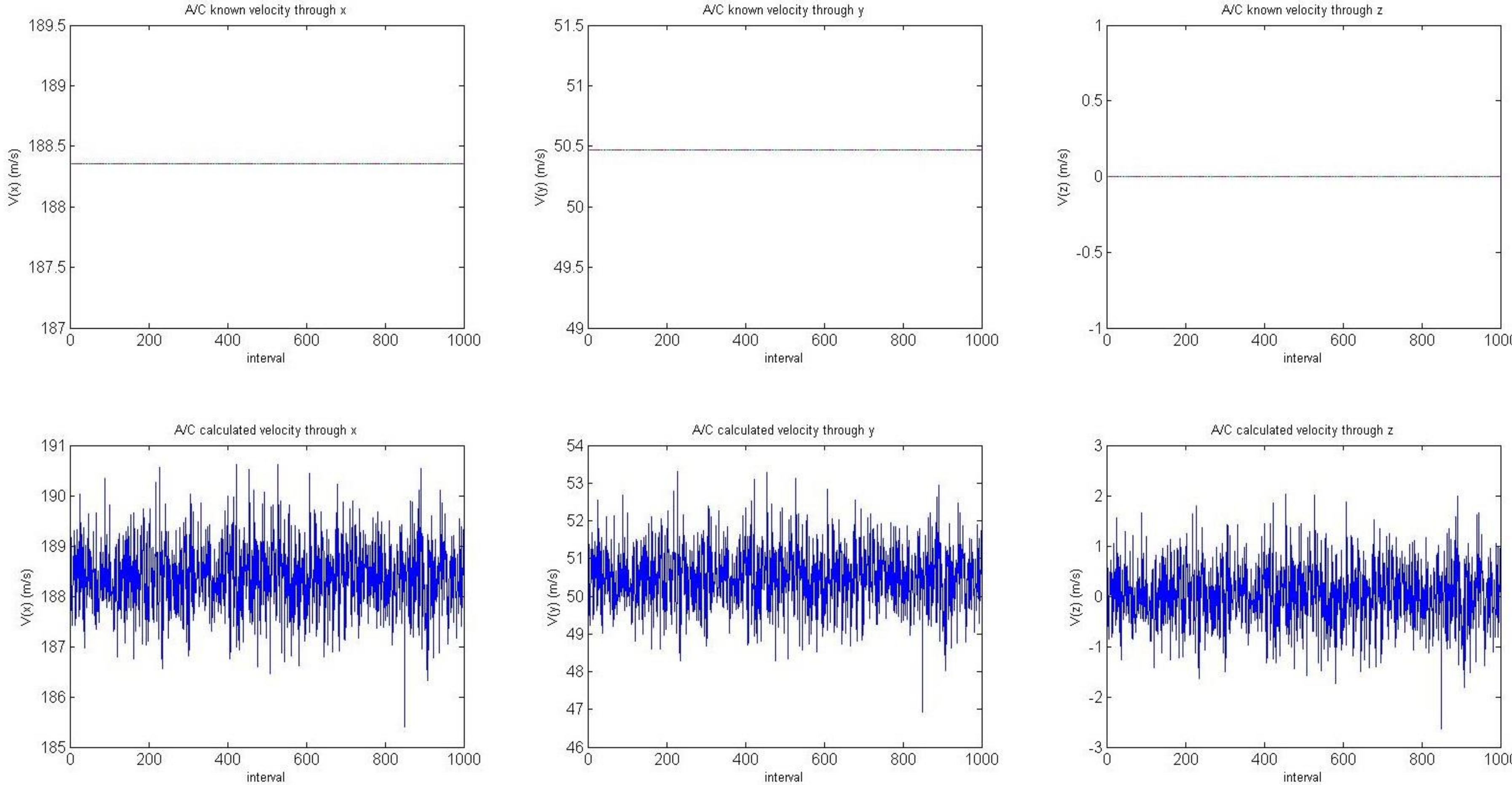

Figure 6.2 : Aircraft reference (simulated) and calculated (via Doppler Method) velocities

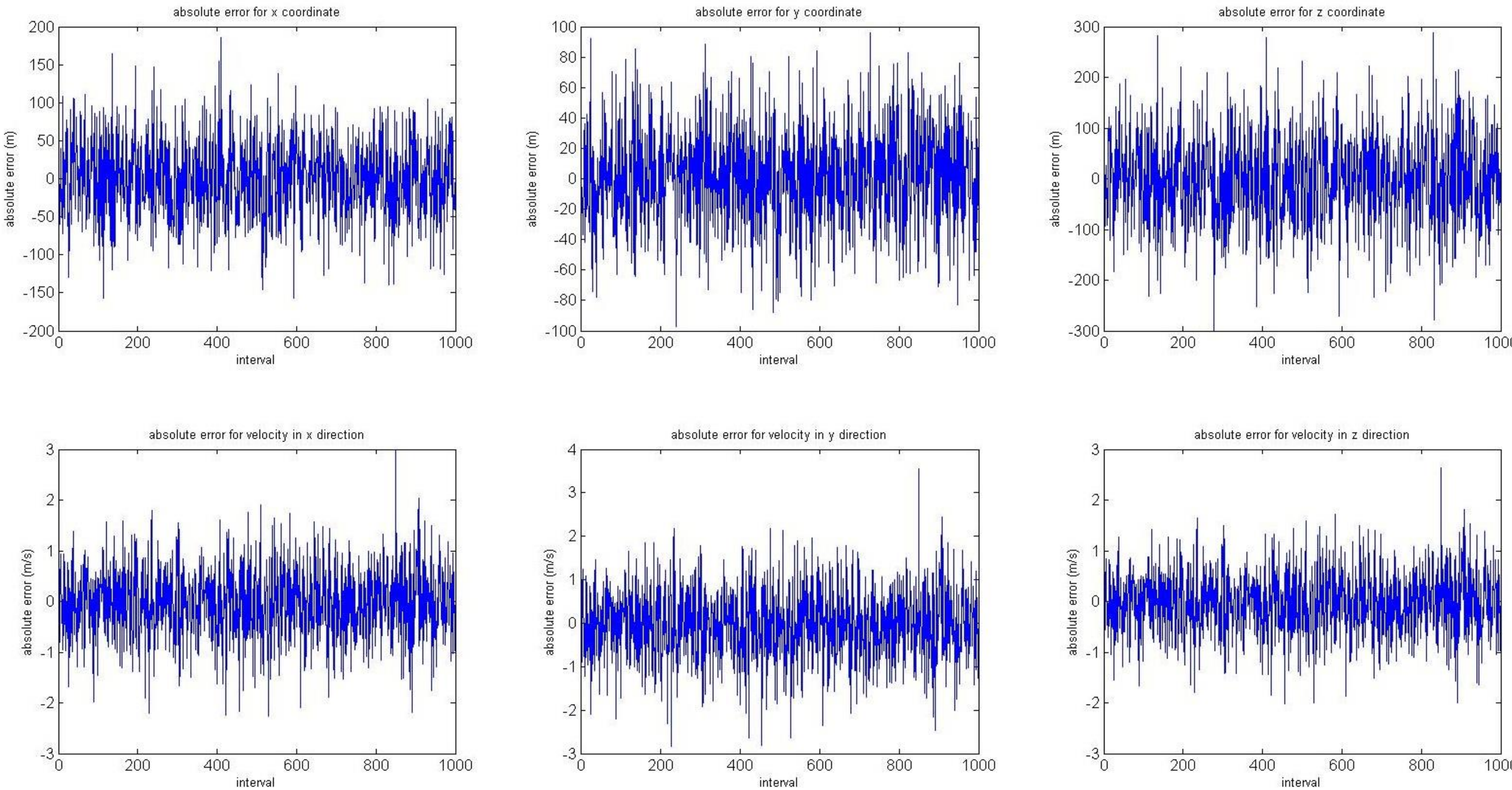

Figure 6.3 : Absolute errors about aircraft position and velocity.

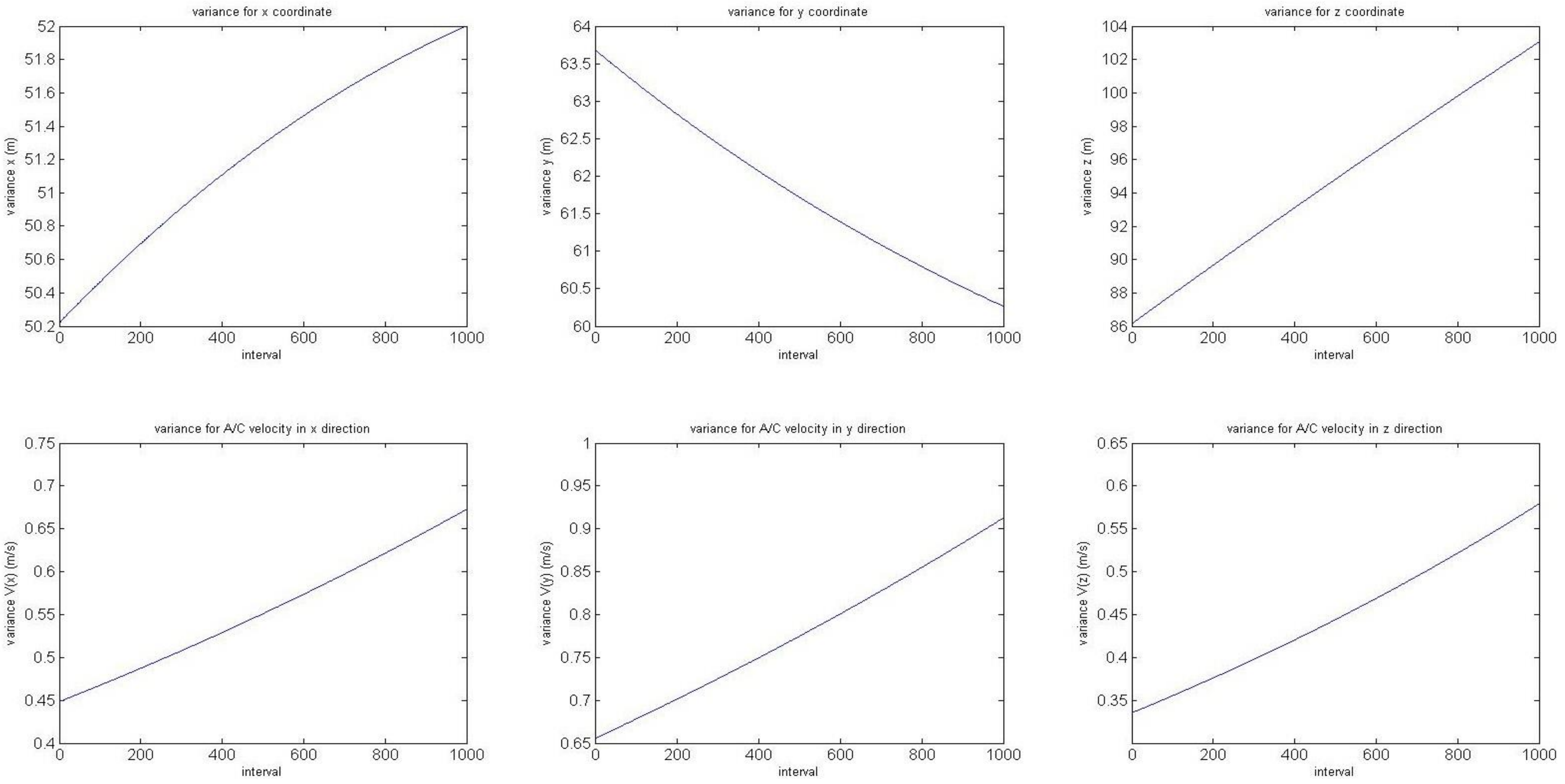

Figure 6.4 : Variances for aircraft position and velocity

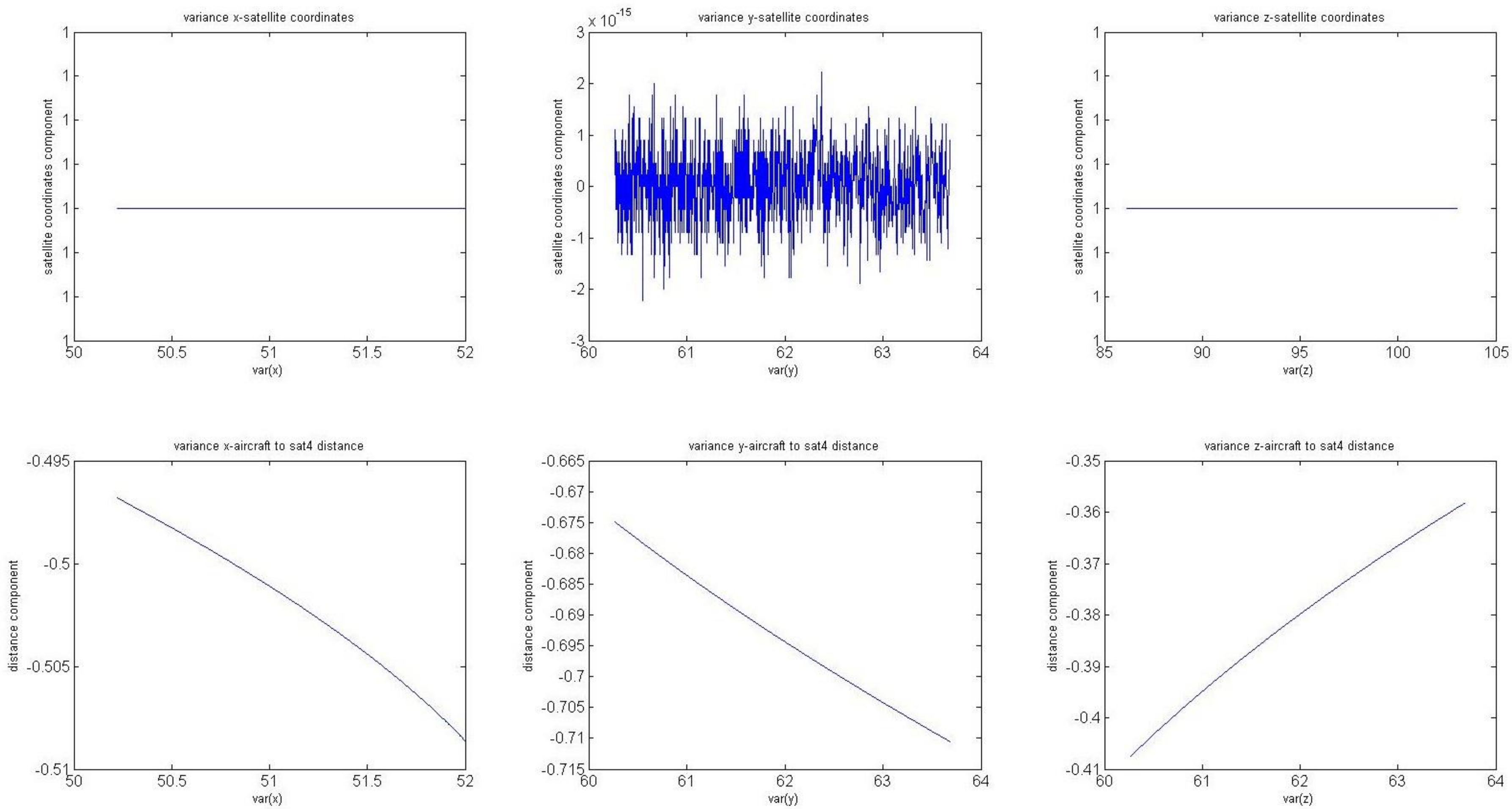

Figure 6.5 : Aircraft position variances versus satellites' coordinates component and distance to router satellite component

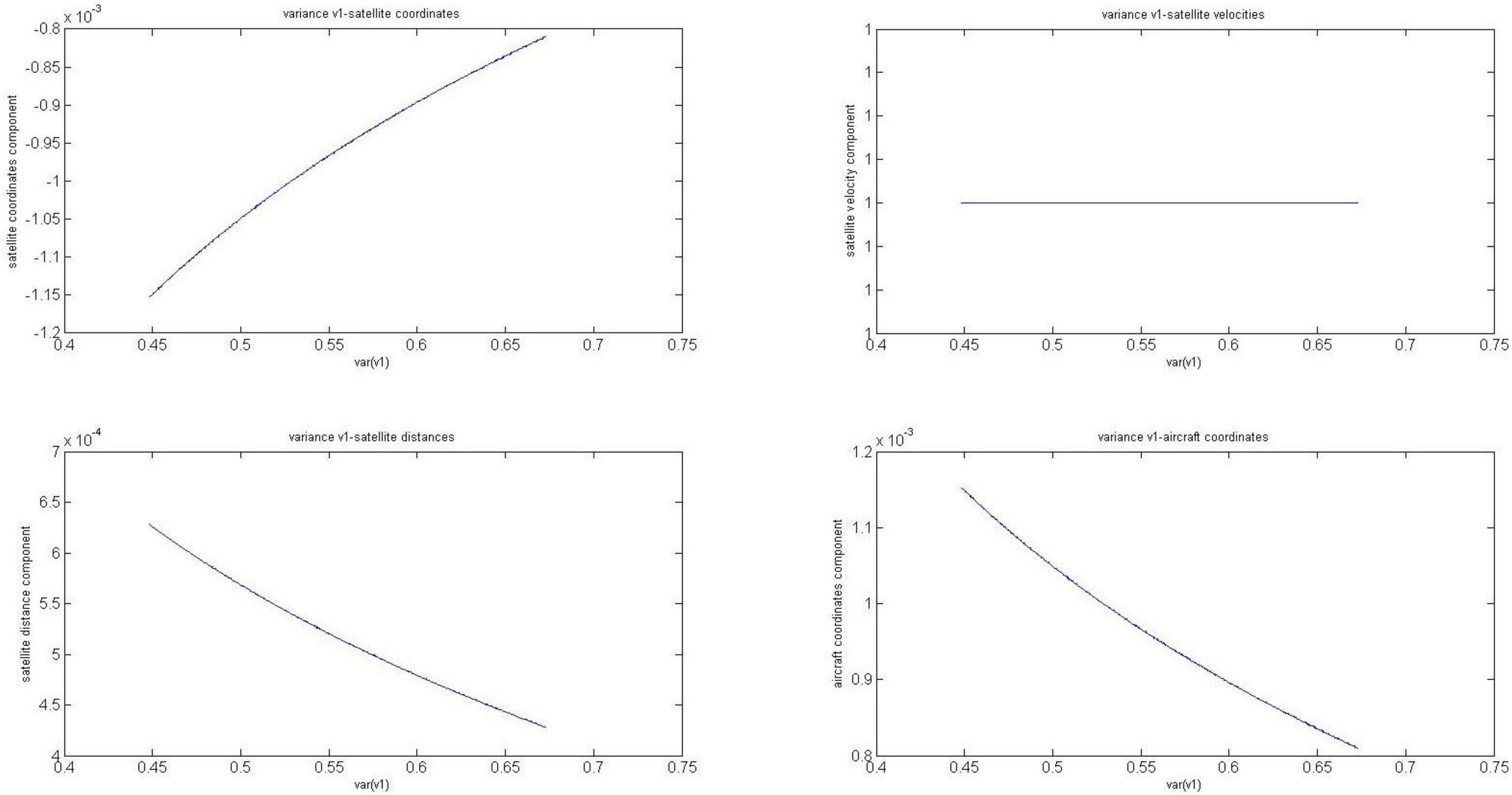

Figure 6.6 : Aircraft “x” velocity variances versus satellites’ coordinates, velocities, distances and aircraft coordinates components

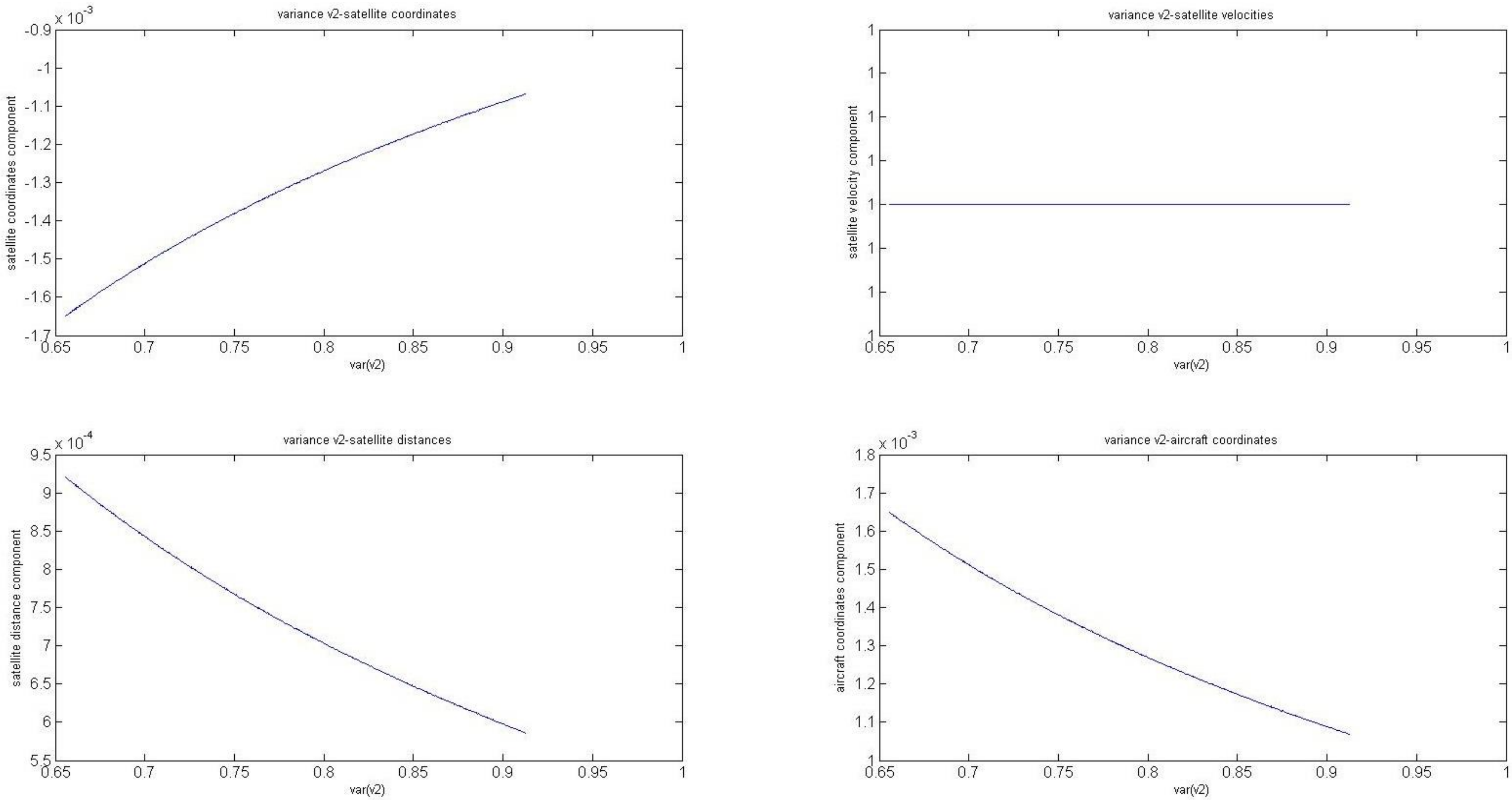

Figure 6.7 : Aircraft “y” velocity variances versus satellites’ coordinates, velocities, distances and aircraft coordinates components

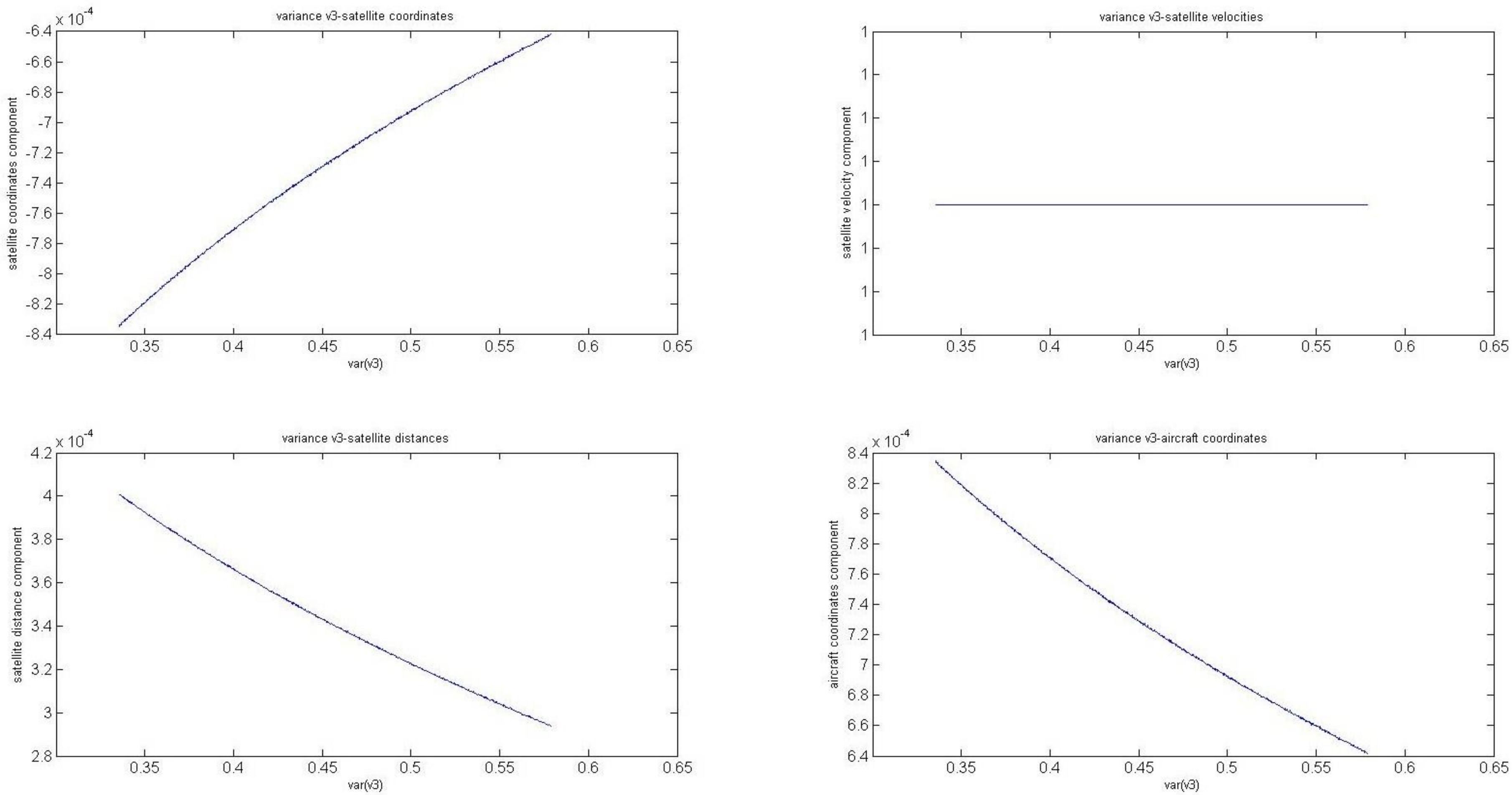

Figure 6.8 : Aircraft “z” variances versus satellites’ coordinates, velocities, distances and aircraft coordinates components velocity

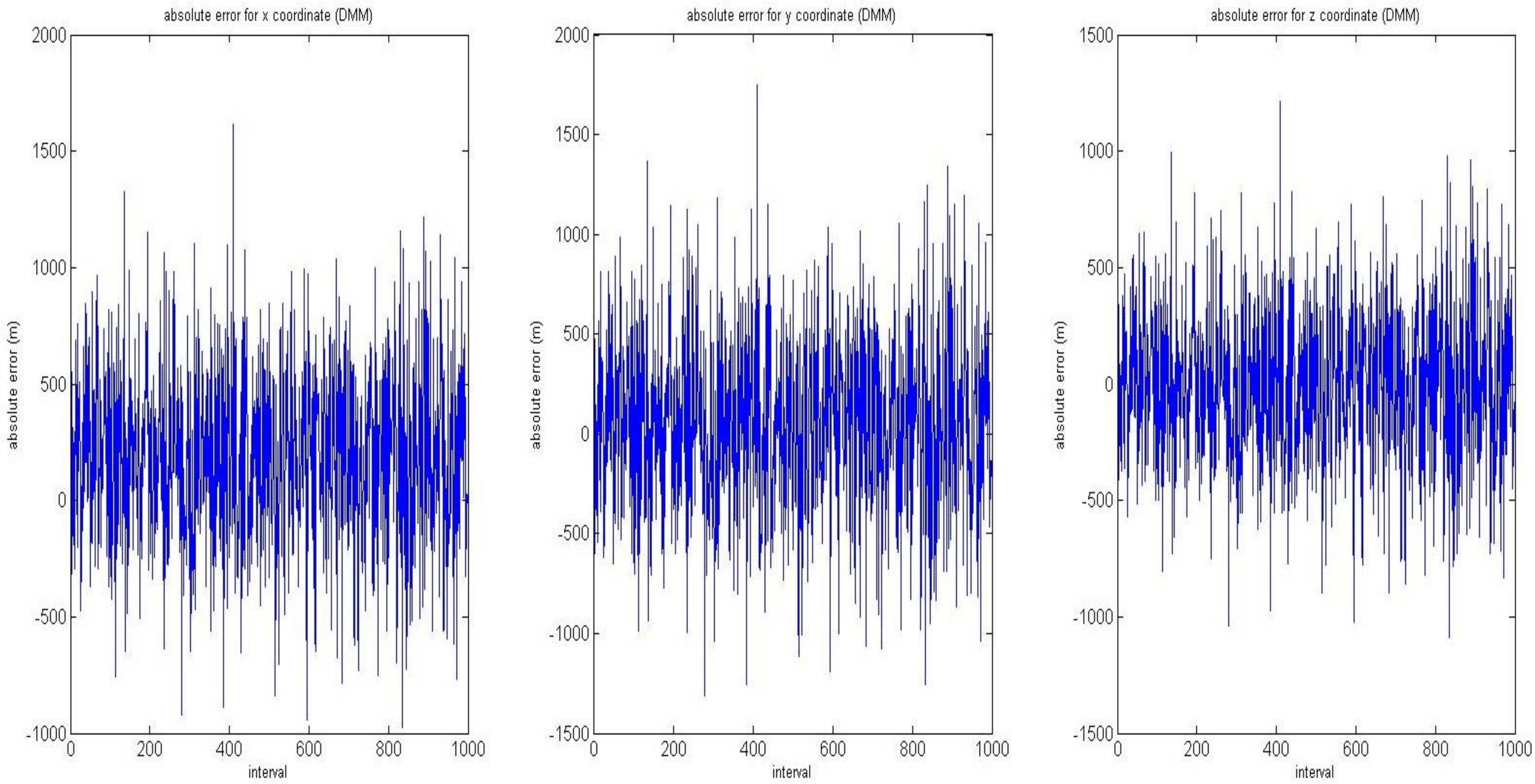

Figure 6.9 : Absolute error of aircraft position via Distance Measuring Method.

CONCLUSION

In the scope of this study, analyses of Distance Difference and Doppler Method were practiced. The aim was to investigate the simplicity of these methods for determining aircraft radio-navigation parameters and to see how the systematical error could be reduced with them as it is mentioned earlier, if the measuring methods can be eliminated from measurement noises, the accuracy is going to rise. It is known that, commercial aircrafts are using the compound method of distance and angle measuring methods, which is because they are eliminating the clock bias effect of GPS satellites via differentially with high accuracy. The methods are offering another precise solution in their own circumstances as differential technique had been used. At the end, Distance Difference and Doppler Method are two good ways of determining the position and velocity.

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