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LINEARIZED STRUCTURAL DYNAMICS MODEL FOR THE SENSITIVITY ANALYSIS OF HELICOPTER ROTOR BLADES

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ABSTRACT

Linearized structural dynamics of helicopter rotor blades and the sensitivity analysis of rotor blade eigenvalues to the critical design variables related to structural dynamics are studied. The main goal is to develop a tool for conceptual aeroservoelastic design of rotorcraft. For this purpose, a linearized structural dynamics model for rotating blades is developed using finite volume method. Using the element matrices of the finite volume model, analytical sensitivities of mass and stiffness matrices to inertia and elasticity parameters are formulated to evaluate the sensitivities of eigenvalues and eigenvectors. The developed tool is verified and the sensitivity analysis of a typical rotor blade is performed by numerical continuation.

INTRODUCTION

Due to the increased sophistication in rotorcraft design requirements and technology, the conventional conceptual design strategies are not sufficient. Therefore higher certainty is needed in conceptual design phase which can be done by combining classical design tools with high-fidelity analysis including comprehensive analysis, computational fluid dynamics and structural analysis [Johnson, 2010]. For this reason design tools that can provide parametric sensitivity analyses are necessary.

An accurate structural dynamics model is essential in analysis and design of helicopter rotor blades. CAMRAD JA, CAMRAD II, MBDyn, UMARC, FLIGHTLAB and DYMORE are among the most widely used ones [Johnson, 2011]. For this purpose a Finite Volume concept is implemented for rotating C^0 beams based on [Ghiringhelli et al., 2000]. The stiffness of each finite volume beam is represented by 6×6 stiffness matrix. The inertial loads are added in a lumped manner. The nonlinear equations are linearized to allow the computation of in-vacuo modes and sensitivity analysis. The formulation is validated against rotating and non-rotating beam results.

Sensitivity analysis is critical in conceptual design phase where the level of uncertainty is high. In fact at the conceptual design phase the optimum solution is searched which requires the evaluation and interpretation of the sensitivity of the design to prescribed design parameters. When higher sensitivity design parameters are focused, it is more likely to converge to an optimum product. On the other hand, when robustness of the design is considered, sensitivity analysis can prevent violating a requirement or approaching to a limit with a small deviation in a variable that possess very high level of sensitivity. Therefore both optimum design and design robustness in rotor blade structural dynamics is believed to benefit from sensitivity analyses that is implemented with a finite volume beam formulation.

When sensitivity or a design objective to design parameters are considered, the behavior of the system to the selected parameters can be analyzed by numerical continuation procedures [Krauskopf et al., 2007]. Once the

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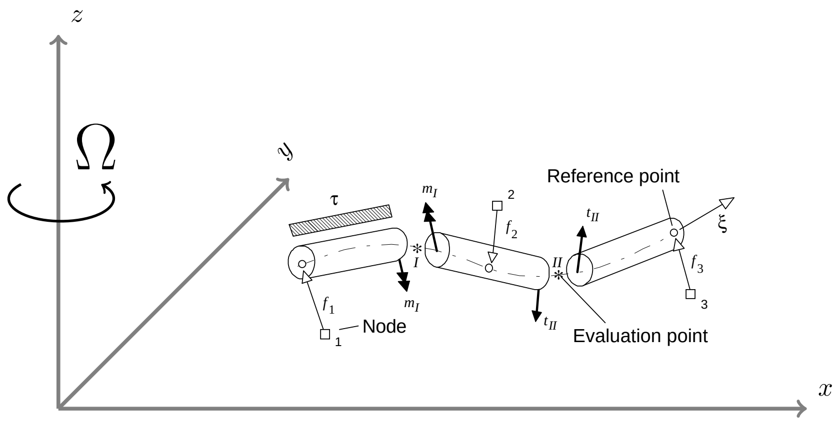


Figure 1: 3-Node Beam element in Global Reference Frame

sensitivity matrices are either analytically developed or evaluated by numerical differentiation, the continuation analysis can provide the directions of the design for optimal performance and the robustness of the system to any change or uncertainty. This is especially important for problems having multiple design parameters.

This paper is organized as follows. The following chapter explains the theory of rotating blade model and then provides the sensitivity relations with respect to the parameters related to cross section inertia and stiffness. In the Numerical analysis section, the tool is verified against well-known tools for rotating and non-rotating natural frequencies. A sensitivity analysis is then performed for a problem which analyzes the required amount of change in design parameters that keeps blade frequencies constant after introducing bending-torsion coupling to a rotating blade.

METHOD

Linearized Elasticity of Blade

Beam Discretization: The structural dynamics model is developed in rotating frame of a single blade based on Finite Volume Formulation [Ghiringhelli et al., 2000]. A three node beam element is used in formulation as given in Figure 1. The longitudinal axis is the blade axis radially outward, the vertical axis (z axis) is parallel to the rotation axis and lateral axis y is the rotor in-plane axis perpendicular to radial axis x and vertical axis z axis. This reference frame is referred to as global frame. The beam reference plane, where elastic properties are defined, are referred to as local reference and lies on a dimensionless coordinate ξ. The variables having an over-tilde $(\tilde{\bullet})$ are defined in this local beam reference. The nodes $(i = 1, 2, 3)$ are the locations where displacements $(\mathbf{x})$ and rotations (φ) about three orthogonal axes are defined. The beam element is defined by three reference points which are attached to three nodes having an offset $\tilde{\mathbf{f}}$ from the corresponding node point in the beam reference frame. Then the location of the beam reference line $(\mathbf{p})$ can be defined as;

$$\mathbf{p} = \mathbf{x} + \mathbf{R}\tilde{\mathbf{f}} \tag{1}$$

where $\mathbf{x}$ is the location of the corresponding node and $\mathbf{R}$ is the rotation matrix of the beam reference frame with respect to global frame. The strain vector $\delta\boldsymbol{\psi}$ is composed of linear strain $\delta\boldsymbol{\varepsilon}$ and linear curvature $\delta\boldsymbol{\kappa}$ in perturbation form. As given in [Ghiringhelli et al., 2000], they can be expressed as;

$$\delta\boldsymbol{\varepsilon} = \delta\mathbf{p}' + \mathbf{p}_0' \times \delta\boldsymbol{\varphi} \;\; \text{and} \;\; \delta\boldsymbol{\kappa} = \delta\boldsymbol{\varphi}' \tag{2}$$

where $\mathbf{p}_0$ is the strain in reference condition and $\delta\boldsymbol{\varphi}$ represents the linearized rotation. In linearized rotation, the rotation can be represented as a vector of three angles about three orthogonal axes. The strains are computed at the evaluation points $(k = I, II)$ as shown in Figure 1. For this beam model the evaluation points are $\xi_I = -1/\sqrt{3}$ and $\xi_{II} = 1/\sqrt{3}$. These points are Gauss quadrature points for polynomials up to third order and they are proven to provide exact behavior for end applied nodes [Ghiringhelli et al., 2000]. The spatial derivative of beam reference line as a function of the nodes are evaluated using second order polynomials (N).

$$N_1(\xi) = \frac{\xi(\xi-1)}{2} \qquad N_2(\xi) = 1 - \xi^2 \qquad N_3(\xi) = \frac{\xi(\xi+1)}{2} \tag{3}$$

where $\xi = -1, 0, 1$ are the points on local coordinate (ξ) corresponding to start, mid and end nodes respectively. The position and rotation vector and their derivatives at local coordinate (ξ) can be expressed as a summation of all three beam reference points for $i = 1, 3$;

$$\mathbf{p}(\xi) = N_i(\xi)(\mathbf{x}_i + \mathbf{R}_i\tilde{\mathbf{f}}_i) \ \ , \ \ \mathbf{p}'(\xi) = N_i'(\xi)(\mathbf{x}_i + \mathbf{R}_i\tilde{\mathbf{f}}_i) \ \text{ and } \ \boldsymbol{\varphi}(\xi) = N_i(\xi)\boldsymbol{\varphi}_i \ \ , \ \ \boldsymbol{\varphi}'(\xi) = N_i'(\xi)\boldsymbol{\varphi}_i \tag{4}$$

Equilibrium Equation and Linearization: The rotating blade is discretized by using a C^0 beam, meaning that displacement and rotation are two separate fields, based on the finite volume concept [Ghiringhelli et al., 2000]. The equation of beam can be stated as an equilibrium between internal forces and moment ($\boldsymbol{\vartheta}$) and distributed forces and moments ($\boldsymbol{\tau}$);

$$\boldsymbol{\vartheta}' - \mathbf{T}^T\boldsymbol{\vartheta} + \boldsymbol{\tau} = 0 \tag{5}$$

where derivation is evaluated with respect to beam reference line ξ and $\mathbf{T}^T$ is the arm matrix of internal forces.

$$\mathbf{T} = \begin{bmatrix} 0 & \mathbf{p}'\times \\ 0 & 0 \end{bmatrix} \tag{6}$$

The internal loads ($\boldsymbol{\vartheta} = [\boldsymbol{\vartheta}_I^T \ \ \boldsymbol{\vartheta}_{II}^T]^T$) evaluated at the evaluation points and the external loads ($\mathbf{F} = [\mathbf{F}_1^T \ \ \mathbf{F}_2^T \ \ \mathbf{F}_3^T]^T$) applied at node points are in equilibrium according to the formulation [Ghiringhelli et al., 2000];

$$\mathbf{A}\boldsymbol{\vartheta} = \mathbf{F} \tag{7}$$

where $\mathbf{A}$ is the integrated arms matrix (Eq. 6) which guarantees equilibrium between the external loads at the nodes and internal loads at the evaluation points.

$$\mathbf{A} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ (\mathbf{p}_I - \mathbf{x}_1)\times & -\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ -(\mathbf{p}_I - \mathbf{x}_2)\times & \mathbf{I} & (\mathbf{p}_{II} - \mathbf{x}_1)\times & -\mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -(\mathbf{p}_{II} - \mathbf{x}_3)\times & \mathbf{I} \end{bmatrix} \tag{8}$$

If perturbation is applied to Eq. 7,

$$\cancel{\mathbf{A}_0\boldsymbol{\vartheta}_0} + \mathbf{A}_0\delta\boldsymbol{\vartheta} + \delta\mathbf{A}\boldsymbol{\vartheta}_0 + \cancel{\delta\mathbf{A}\delta\boldsymbol{\vartheta}} = \cancel{\mathbf{F}_0} + \delta\mathbf{F} \Rightarrow \mathbf{A}_0\delta\boldsymbol{\vartheta} + \delta\mathbf{A}\boldsymbol{\vartheta}_0 = \delta\mathbf{F} \tag{9}$$

where terms with 0 subscript are reference conditions. Eq. 9 is the linearized equilibrium equation around a reference condition after neglecting higher order terms and canceling reference condition. The term $\mathbf{A}_0\delta\boldsymbol{\vartheta}$ gives the perturbation of internal loads with a arm matrix at reference configuration whereas $\delta\mathbf{A}\boldsymbol{\vartheta}_0$ term is responsible from stiffness contribution due to pre-stress of the beam element under reference centrifugal loads.

Constitutive Law: The internal loads $\boldsymbol{\vartheta}$ are related to strains $\boldsymbol{\psi}$ by a linear elastic constitutive law $\boldsymbol{\vartheta}_k = \mathbf{D}_k\boldsymbol{\psi}_k$ at two evaluation points $\xi_I = -1/\sqrt{3}$ and $\xi_{II} = 1/\sqrt{3}$. For an isotropic cross section with double symmetry, the 6×6 material matrix $\mathbf{D}$ about its principal axis has the following form;

$$\mathbf{D} = \begin{bmatrix} EA & 0 & 0 & 0 & 0 & 0 \\ 0 & GA_y & 0 & 0 & 0 & 0 \\ 0 & 0 & GA_z & 0 & 0 & 0 \\ 0 & 0 & 0 & GJ & 0 & 0 \\ 0 & 0 & 0 & 0 & EI_y & 0 \\ 0 & 0 & 0 & 0 & 0 & EI_z \end{bmatrix} \tag{10}$$

where EA is the axial stiffness, GA_y and GA_z are the shear stiffnesses about shear principal axes, GJ is the torsional stiffness, EJ_y and EJ_z are the bending stiffnesses about bending principal axes. For an orthotropic cross section, $\mathbf{D}$ can be fully populated. In general, accurate beam constitutive properties for non-homogeneous and anisotropic sections can be formulated using the approach originally proposed by [Giavotto et al., 1983].

Perturbation of Internal Loads: Considering that it is more convenient to provide material matrix in beam reference frame, internal forces given can be expressed in global frame as ;

$$\boldsymbol{\vartheta} = \boldsymbol{\mathcal{D}}\boldsymbol{\psi} = \boldsymbol{\mathcal{R}}\tilde{\boldsymbol{\mathcal{D}}}\boldsymbol{\mathcal{R}}^T\boldsymbol{\psi} \tag{11}$$

$\boldsymbol{\mathcal{R}}$ and $\tilde{\boldsymbol{\mathcal{D}}}$ are the rotation and constitutive law matrices having the contributions from two evaluation points and similarly $\boldsymbol{\vartheta}$ and $\boldsymbol{\psi}$ are the internal load and strain vectors at the evaluation points.

$$\boldsymbol{\mathcal{R}} = diag([\mathbf{R}_I \ \ \mathbf{R}_I \ \ \mathbf{R}_{II} \ \ \mathbf{R}_{II}]) \ \ , \ \ \tilde{\boldsymbol{\mathcal{D}}} = diag([\tilde{\mathbf{D}}_I \ \ \tilde{\mathbf{D}}_{II}]) \ \ , \ \ \boldsymbol{\vartheta} = [\boldsymbol{\vartheta}_I^T \ \ \boldsymbol{\vartheta}_{II}^T]^T \ \ , \ \ \boldsymbol{\psi} = [\boldsymbol{\psi}_I^T \ \ \boldsymbol{\psi}_{II}^T]^T \tag{12}$$

Then the perturbation of internal loads $\boldsymbol{\vartheta}$ is;

$$\delta\boldsymbol{\vartheta} = \delta\boldsymbol{\mathcal{R}}\tilde{\boldsymbol{\mathcal{D}}}\boldsymbol{\mathcal{R}}_0^T\boldsymbol{\psi}_0 + \boldsymbol{\mathcal{R}}_0\tilde{\boldsymbol{\mathcal{D}}}\delta\boldsymbol{\mathcal{R}}^T\boldsymbol{\psi}_0 + \boldsymbol{\mathcal{R}}_0\tilde{\boldsymbol{\mathcal{D}}}\boldsymbol{\mathcal{R}}^T\delta\boldsymbol{\psi} \tag{13}$$

where terms with 0 index represents reference conditions. Remembering that $\delta\mathbf{R} = \delta\boldsymbol{\varphi}\times\mathbf{R}_0$ and $\delta\mathbf{R}^T = \mathbf{R}_0^T\delta\boldsymbol{\varphi}\times$, for a rotation matrix $\mathbf{R}$ and corresponding linearized rotation angle $\delta\boldsymbol{\varphi}$, Eq. 13 is rearranged as,

$$\delta\boldsymbol{\vartheta} = -\boldsymbol{\vartheta}_0\times\delta\boldsymbol{\varphi} + \boldsymbol{\mathcal{R}}_0\boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{R}}_0^T(\delta\boldsymbol{\psi} + \boldsymbol{\psi}_0\times\delta\boldsymbol{\varphi}) \tag{14}$$

where $\boldsymbol{\vartheta}_0$ and $\boldsymbol{\psi}_\mathbf{0}$ are the reference internal forces and strains and $\times$ represents the matrix from of a vector product operation. Considering that in helicopter blades, centrifugal loads are significantly larger then other form of external loads, only tension forces due to rotation ($\mathbf{t_0}$) at evaluation points are considered as reference internal loads. Then $\boldsymbol{\vartheta}_0\times$ matrix becomes;

$$\boldsymbol{\vartheta}_0\times = \begin{bmatrix} \mathbf{0} & \mathbf{t}_{I0}\times & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{t}_{II0}\times \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{15}$$

Finally $\delta\boldsymbol{\psi}$ can be written using shape functions for node indices $i = 1,3$, with $\mathbf{f} = \mathbf{R}\tilde{\mathbf{f}}$, where $\tilde{\mathbf{f}}$ is the offset in node reference frame.

$$\delta\boldsymbol{\psi} = \begin{bmatrix} \delta\boldsymbol{\varepsilon}_I \\ \delta\boldsymbol{\kappa}_I \\ \delta\boldsymbol{\varepsilon}_{II} \\ \delta\boldsymbol{\kappa}_{II} \end{bmatrix} = \begin{bmatrix} N'_{Ii}\mathbf{I} & \mathbf{p}'_{0I}\times N_{Ii} - N'_{Ii}\mathbf{f}_i\times \\ 0 & N'_{I,i}\mathbf{I} \\ N'_{IIi}\mathbf{I} & \mathbf{p}'_{0I}\times N_{IIi} - N'_{IIi}\mathbf{f}_i \\ 0 & N'_{IIi}\mathbf{I} \end{bmatrix} \begin{bmatrix} \delta\mathbf{x}_i \\ \delta\boldsymbol{\varphi}_i \end{bmatrix} \tag{16}$$

Stiffness contribution due to pre-stress: The $\delta\mathbf{A}\boldsymbol{\vartheta}_\mathbf{0}$ term in Eq. 9 is the effect of reference loads on the response of the beam which can be considered as stiffening due to reference loads. The linearization of arms matrix $\mathbf{A}$ comes from Eq. 6 by perturbing the variables $\mathbf{p}$ and $\mathbf{x}$. This matrix is multiplied by tension due to centrifugal loads. Without going into detail, collecting the variables as a right hand side vector, the stiffness contribution due to pre-stress can be obtained as;

$$\mathbf{S_P} = \delta\mathbf{A}\boldsymbol{\vartheta}_0 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{t}_{I0}\times(N_{Ii}-\delta_{i1}) & -\mathbf{t}_{I0}\times N_{Ii}\mathbf{f}_i\times \\ \mathbf{0} & \mathbf{0} \\ \begin{pmatrix} -\mathbf{t}_{I0}\times(N_{Ii}-\delta_{i2}) \\ +\mathbf{t}_{II0}\times(N_{IIi}-\delta_{i2}) \end{pmatrix} & \begin{pmatrix} \mathbf{t}_{I0}\times N_{Ii}\mathbf{f}_i\times \\ -\mathbf{t}_{II0}\times N_{IIi}\mathbf{f}_i\times \end{pmatrix} \\ \mathbf{0} & \mathbf{0} \\ -\mathbf{t}_{II0}\times(N_{IIi}-\delta_{i3}) & \mathbf{t}_{II0}\times N_{IIi}\mathbf{f}_i\times \end{bmatrix} \begin{bmatrix} \delta\mathbf{x}_i \\ \delta\boldsymbol{\varphi}_i \end{bmatrix} \tag{17}$$

where δ_{ij} is the Kronecker delta operator.

Inertial Loads

The inertial loads are taken into account under the external loads ($\mathbf{F}$) given in Eq. 7. An exact representation requires integration over the beam element. However, using lumped mass and inertia gives enough accuracy without increasing computational work. Therefore inertial loads are formalized by using lumped masses and inertias for a constant rotor angular speed.

Lumped Mass and Inertia in Rotating Frame: The main strategy of lumped mass and inertia application is to discretize the blade similar to that of elastic beam elements. Since a beam element is composed of three sections in elastic discretization, it is also decided to represent the lumped masses and moments at the same sections as given in Figure 2 where spheres represents the lumped mass having inertia. The inertial force acting on a node can be represented as;

$$\mathbf{F_{in}} = -m\frac{d^2}{dt^2}(\mathbf{x} + \mathbf{f_{CM}}) \tag{18}$$

where the center of mass in global reference frame is given as $\mathbf{f_{CM}} = \mathbf{R}\tilde{\mathbf{f}}_\mathbf{CM}$ with center of mass offset $\tilde{\mathbf{f}}_\mathbf{CM}$ given in beam reference frame. The load in rotating reference frame can be expressed after differentiation as;

$$\mathbf{F_{in}} = -m(\ddot{\mathbf{x}} + \dot{\boldsymbol{\omega}}\times\mathbf{f_{CM}} + \boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{f_{CM}}) \tag{19}$$

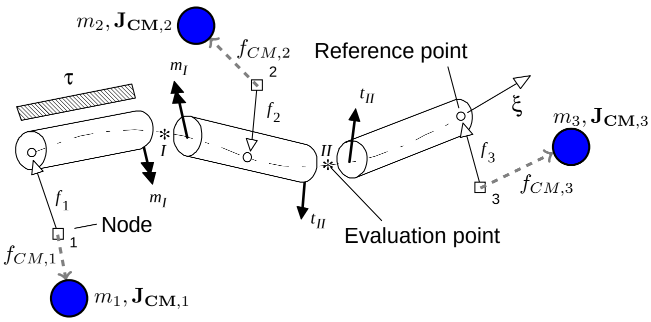


Figure 2: Lumped mass and inertia

where the term $\ddot{\mathbf{x}}$ and $\dot{\boldsymbol{\omega}}$ are the linear and rotational accelerations of the node in rotating reference frame. The higher order term $(\boldsymbol{\omega} \times \boldsymbol{\omega}\times)$ can be neglected since the code deals with the perturbation displacement and rotations. For constant RPM, the inertial loads formula becomes;

$$\mathbf{F_{in}} = -m(\ddot{\mathbf{x}} - \mathbf{f_{CM}} \times \dot{\boldsymbol{\omega}}) = -m(\ddot{\mathbf{x}} + \mathbf{f_{CM}} \times^T \dot{\boldsymbol{\omega}}) \tag{20}$$

The moment at the node comes from two contributions. The first one is the derivative of angular momentum; the other one is the force contribution that originates from offset of center of mass, which can be written as;

$$\mathbf{M_{in}} = -\mathbf{J_{CM}}\dot{\boldsymbol{\omega}} - \boldsymbol{\omega} \times \mathbf{J_{CM}}\boldsymbol{\omega} + \mathbf{f_{CM}} \times \mathbf{F_{in}} \tag{21}$$

where $\mathbf{J_{CM}}$ is the inertia matrix at the center of mass. Inserting inertial force into above equation;

$$\mathbf{M_{in}} = -\mathbf{J_{CM}}\dot{\boldsymbol{\omega}} - \boldsymbol{\omega} \times \mathbf{J_{CM}}\boldsymbol{\omega} - m(\mathbf{f_{CM}} \times \ddot{\mathbf{x}} - \mathbf{f_{CM}} \times \mathbf{f_{CM}} \times \dot{\boldsymbol{\omega}}) \tag{22}$$

Following the same procedure that was applied to inertial force and defining inertia about node $\mathbf{J_n}$ as $\mathbf{J_n} = \mathbf{J_{CM}} + m\mathbf{f_{CM}} \times \mathbf{f_{CM}}\times^T$ leads to compact moment equation;

$$\mathbf{M_{in}} = -\mathbf{J_n}\dot{\boldsymbol{\omega}} - m\mathbf{f_{CM}} \times \ddot{\mathbf{x}} = -\mathbf{J_n}\dot{\omega} + m\mathbf{f_{CM}} \times^T \ddot{\mathbf{x}} \tag{23}$$

Remembering that for small angles the angular velocity is the time derivation of the rotation angle, i.e $\boldsymbol{\omega} = \dot{\boldsymbol{\varphi}}$ and writing force and moment equations in perturbation form and collecting in matrix form yields

$$\begin{bmatrix} \delta\mathbf{F_{in}} \\ \delta\mathbf{M_{in}} \end{bmatrix} = -\begin{bmatrix} m\mathbf{I} & m\mathbf{f_{CM}}\times^T \\ m\mathbf{f_{CM}}\times & \mathbf{J_n} \end{bmatrix} \begin{bmatrix} \delta\ddot{\mathbf{x}} \\ \delta\ddot{\boldsymbol{\varphi}} \end{bmatrix} \tag{24}$$

<u>Effect of Rotation:</u> The lumped mass formulation in previous chapter is given for a mass in rotating reference frame. However in order to see the effect of rotation, the acceleration in rotating reference frame $\ddot{\mathbf{x}}$ should be decomposed into its constituents. The transformation matrix from non-rotating reference frame variable $\mathbf{x}_{nr}$ to rotating reference frame variable $\mathbf{x}_r$ can be written as

$$\mathbf{x}_{nr} = \mathbf{R}_\Omega \mathbf{x}_r \tag{25}$$

where $\mathbf{R_\Omega}$ is the transformation from rotating to non-rotating reference frame. Differentiating once gives velocity in non rotating frame,

$$\dot{\mathbf{x}}_{nr} = \boldsymbol{\Omega} \times \mathbf{R}_\Omega \mathbf{x}_r + \mathbf{R_\Omega}\dot{\mathbf{x}}_r \tag{26}$$

where derivative of orientation vector for a rotating frame is $\dot{\mathbf{R}} = \boldsymbol{\Omega} \times \mathbf{R}$. Differentiating once more gives acceleration in non rotating frame,

$$\ddot{\mathbf{x}}_{nr} = (\dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega} \times \boldsymbol{\Omega}\times)\mathbf{R}_\Omega \mathbf{x}_r + 2\boldsymbol{\Omega} \times \mathbf{R}_\Omega \dot{\mathbf{x}}_r + \mathbf{R}_\Omega \ddot{\mathbf{x}}_r \tag{27}$$

The acceleration in non-rotating reference frame is required to be projected to rotating frame in which loads acting on blades are expressed. Considering that for a rotation matrix $\mathbf{R}\mathbf{R}^T = \mathbf{I}$, when Eq. 25 is multiplied by $\mathbf{R}_\Omega^T$ it can be stated as $\mathbf{x}_r = \mathbf{R}_\Omega^T \mathbf{x}_{nr}$. Then the acceleration projected to rotating reference frame becomes;

$$\mathbf{R}_\Omega^T \ddot{\mathbf{x}}_{nr} = (\mathbf{R}_\Omega^T \dot{\boldsymbol{\Omega}} \times \mathbf{R}_\Omega + \mathbf{R}_\Omega^T \boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \mathbf{R}_\Omega)\mathbf{x}_r + 2\mathbf{R}_\Omega^T \boldsymbol{\Omega} \times \mathbf{R}_\Omega \dot{\mathbf{x}}_r + \mathbf{R}_\Omega^T \mathbf{R}_\Omega \ddot{\mathbf{x}}_r \tag{28}$$

Since $\mathbf{R}_\Omega$ is the rotation matrix caused by the angular velocity $\mathbf{\Omega}$ about a single axis, $\mathbf{R}_\Omega^T\mathbf{\Omega} = \mathbf{\Omega}$ and $\mathbf{R}_\Omega^T\mathbf{\Omega} = \mathbf{\Omega}$ and $\mathbf{R}_\Omega^T\mathbf{\Omega} \times \mathbf{R}_\Omega = (\mathbf{R}_\Omega^T\mathbf{\Omega})\times$. Then the acceleration in non-rotating frame can be projected on rotating reference frame as

$$\mathbf{R}_\Omega^T\ddot{\mathbf{x}}_{nr} = (\dot{\mathbf{\Omega}} \times + \mathbf{\Omega} \times \mathbf{\Omega}\times)\mathbf{x}_r + 2\mathbf{\Omega} \times \dot{\mathbf{x}}_r + \ddot{\mathbf{x}}_r \tag{29}$$

Moreover since the rotor speed is held constant and $\mathbf{x} = \mathbf{x}_r$ by definition, the equation reduces to

$$\mathbf{R}_\Omega^T\ddot{\mathbf{x}}_{nr} = \mathbf{\Omega} \times \mathbf{\Omega} \times \mathbf{x} + 2\mathbf{\Omega} \times \dot{\mathbf{x}} + \ddot{\mathbf{x}} \tag{30}$$

The $\ddot{\mathbf{x}}$ term in Eq. 30 is already included in Lumped Mass formula and gives usual mass matrix in the rotating frame. The rotation effects comes from the remaining terms which are explained in next two paragraphs.

Destabilization Force: The $\mathbf{\Omega} \times \mathbf{\Omega} \times \mathbf{x}$ term in Eq. 30 is the steady centrifugal force acting on the blade section which is included before in the stiffness contribution due to pre-stress and responsible for the centrifugal stiffening. The perturbation due to this term gives additional term that is important in rotor dynamics;

$$\delta\mathbf{F}_{\boldsymbol{DS}} = -m\mathbf{\Omega} \times \mathbf{\Omega} \times \delta\mathbf{x} \tag{31}$$

When angular velocity is taken about z axis $\mathbf{\Omega} = [0 \;\; 0 \;\; \Omega_z]^T$ and the perturbations of node in x, y and z directions given in the rotating reference frame can be stated as $\delta x = [\delta u \;\; \delta v \;\; \delta w]^T$. The perturbation in x direction can be neglected since external perturbation loads in that direction is negligible as compared to the other two directions. Then the final form of the equation can be written as;

$$\delta F_{DS} = m\Omega_z^2\delta v_r \tag{32}$$

This force is referred to as destabilization force since as opposed to Eq. 24, the sign is positive therefore ads energy into system. The effect of this force is in the lead lag direction due to the δv term.

Coriolis Forces: The $2\Omega \times \dot{x}_r$ term in Eq. 30 gives additional term that is important in rotor dynamics. The perturbation force due to this term gives Coriolis Forces which is discarded in normal modes analysis.

$$\delta\mathbf{F}_{\boldsymbol{cor}} = -m2\mathbf{\Omega} \times \delta\dot{\mathbf{x}} \tag{33}$$

Continuation

General form of free vibration equation is written as a function of eigenvalues ($s = \mathrm{j}\omega$) and eigenvector $\mathbf{x}(s)$;

$$s^2\mathbf{M}\mathbf{x}(s) + \mathbf{K}\mathbf{x}(s) = 0 \tag{34}$$

where $\mathbf{M}$ and $\mathbf{K}$ are the global mass and stiffness matrices of the structural dynamics model. These matrices are obtained by assembling the element matrices as described in previous section.

For sensitivity purposes, the change of the solution with respect to a parameter (p) is looked for. Hence all the matrices ($\mathbf{M} = \mathbf{M}(p), \mathbf{K} = \mathbf{K}(p)$), eigenvalue ($s = s(p)$) and eigenvector ($\mathbf{x} = \mathbf{x}(p)$) of Eq. 34 are functions of that parameter. Then differentiating with respect to parameter p;

$$s^2\mathbf{M}_{/p}\mathbf{x} + s^2\mathbf{M}\mathbf{x}_{/p} + 2ss_{/p}\mathbf{M}\mathbf{x} + \mathbf{K}_{/p}\mathbf{x} + \mathbf{K}\mathbf{x}_{/p} = 0 \tag{35}$$

where s and $\mathbf{x}(s)$ is a solution of Eq. 34 and terms with subscript $/p$ represent derivative with respect to the prescribed parameter p. The mass and stiffness sensitivity matrices, $\mathbf{M}_{/p}$ and $\mathbf{K}_{/p}$, are evaluated at that solution point. The derivatives of the matrices are assumed to be known at this point and in subsequent sections they are provided for a set of design variables. If the number of degrees of freedom is n, then for each eigenvalue s, x gives a vector having a dimension of n. On the other hand, the unknowns are the derivatives of eigenvalue ($s_{/p}$) and eigenvector ($\mathbf{x}_{/p}$), a total of $n + 1$. Therefore Eq. 35 provides n equations whereas there are $n + 1$ unknowns since both of ($s_{/p}$) and ($\mathbf{x}_{/p}$) are needed to be evaluated. One more equation is required and it comes from the orthogonality of the normalized eigenvector;

$$\mathbf{x}^T\mathbf{x} = 1 \;\; \text{then} \;\; 2\mathbf{x}^T\mathbf{x}_{/p} = 0 \tag{36}$$

Now the equation is full rank, provided the multiplicity of the eigenvalue is 1 and hence the stability derivatives can be evaluated. Combining Eq. 35 and Eq. 36 in matrix form gives the first order derivatives of eigenvalue ($s_{/p}$) and eigenvector ($x_{/p}$) which can be evaluated using the matrix form.

$$\begin{bmatrix} s^2\mathbf{M} + \mathbf{K} & 2s\mathbf{M} \\ 2\mathbf{x}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_{/p} \\ s_{/p} \end{bmatrix} = \begin{bmatrix} -(s^2\mathbf{M}_{/p} + \mathbf{K}_{/p}) \\ 0 \end{bmatrix} \tag{37}$$

Sensitivity Matrices

The sensitivity of mass and stiffness matrices are formulated in element level. The element matrices are then assembled to obtain global sensitivity matrices. Since this work focuses on structural dynamics, the sensitivities with respect to mass, torsional inertia and stiffness elements of constitutive law are derived.

Constitutive Law: The constitutive law is a 6×6 matrix as mentioned before. It can depend on any parameter of the blade section, from the cross section dimensions such as chord and thickness to the material properties. It should be noted that the cross section is defined in beam reference frame so that the sensitivity is independent of the orientation of the cross section.

The constitutive law matrix $\tilde{\mathbf{D}}$ appears in the perturbation of the internal loads, $\delta\boldsymbol{\vartheta}$, given in Eq. 14. Assuming that that the effect of reference strain $\boldsymbol{\psi}_0$ is negligible, the perturbation of inertial loads becomes;

$$\delta\boldsymbol{\vartheta} = -\boldsymbol{\vartheta}_0 \times \delta\boldsymbol{\varphi} + \boldsymbol{\mathcal{R}}_0 \tilde{\boldsymbol{\mathcal{D}}} \boldsymbol{\mathcal{R}}_0^T \delta\boldsymbol{\psi} \tag{38}$$

Differentiation with respect to a parameter p which is a variable of the constitutive law matrix $\tilde{\boldsymbol{\mathcal{D}}}$ gives;

$$\delta\boldsymbol{\vartheta}_{/p} = \boldsymbol{\mathcal{R}}_0 \tilde{\boldsymbol{\mathcal{D}}}_{/p} \boldsymbol{\mathcal{R}}_0^T \delta\boldsymbol{\psi} \tag{39}$$

where $\tilde{\boldsymbol{\mathcal{D}}}_{/p}$ is required to be provided according to the physical design parameter. If the design variable is one of the stiffness parameters given in Eq. 10, then that element of the matrix is 1 while the rest is null. The assembly of local elements gives the sensitivity of the global stiffness matrix to the prescribed stiffness property.

Mass Distribution: Due to the inertial couplings acting on a rotating blade, the effect of mass distribution is more complex as compared to that of constitutive law matrix and both global mass and global stiffness matrices have sensitivity to mass distribution. For the mass matrix, the sensitivity comes from the inertia matrix of Eq. 24 by differentiating with respect to m,

$$\begin{bmatrix} \delta\mathbf{F}_{\mathbf{in}/p} \\ \delta\mathbf{M}_{\mathbf{in}/p} \end{bmatrix} = -\begin{bmatrix} \mathbf{I} & \mathbf{f}_{\mathbf{CM}} \times^T \\ \mathbf{f}_{\mathbf{CM}} \times & \mathbf{f}_{\mathbf{CM}} \times \mathbf{f}_{\mathbf{CM}} \times^T \end{bmatrix} \begin{bmatrix} \delta\ddot{\mathbf{x}} \\ \delta\ddot{\boldsymbol{\varphi}} \end{bmatrix} \tag{40}$$

The inertial loads also contribute to the sensitivity of stiffness matrix through the destabilizing force given in Eq. 31. The differentiating with respect to m gives,

$$\delta\mathbf{F}_{\mathbf{DS}/p} = -\boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \delta\mathbf{x} \tag{41}$$

Another contribution to sensitivity of stiffness matrix arises from the stiffness contribution due to pre-stress given in Eq. 17 through the reference tension loads $\mathbf{t_{I0}}$, $\mathbf{t_{II0}}$. If these reference loads are evaluated for unit mass $m = 1$, the sensitivity matrix of the stiffness contribution due to pre-stress is obtained.

$$\mathbf{S}_{\mathbf{P}/p} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{t}_{I0,m=1} \times (N_{Ii} - \delta_{i1}) & -\mathbf{t}_{I0,m=1} \times N_{Ii}\mathbf{f}_i \times \\ \mathbf{0} & \mathbf{0} \\ \begin{pmatrix} -\mathbf{t}_{I0,m=1} \times (N_{Ii} - \delta_{i2}) \\ +\mathbf{t}_{II0,m=1} \times (N_{IIi} - \delta_{i2}) \end{pmatrix} & \begin{pmatrix} \mathbf{t}_{I0,m=1} \times N_{Ii}\mathbf{f}_i \times \\ -\mathbf{t}_{II0,m=1} \times N_{IIi}\mathbf{f}_i \times \end{pmatrix} \\ \mathbf{0} & \mathbf{0} \\ -\mathbf{t}_{II0,m=1} \times (N_{IIi} - \delta_{i3}) & \mathbf{t}_{II0,m=1} \times N_{IIi}\mathbf{f}_i \times \end{bmatrix} \begin{bmatrix} \delta\mathbf{x}_i \\ \delta\boldsymbol{\varphi}_i \end{bmatrix} \tag{42}$$

A similar contribution comes from the reference load contribution given in Eq. 31. Again evaluating the reference load for $m = 1$ gives the sensitivity of internal load perturbation to the mass distribution.

$$\delta\boldsymbol{\vartheta}_{/p} = -\boldsymbol{\vartheta}_{0,m=1} \times \delta\boldsymbol{\varphi} \tag{43}$$

Moment of inertia about radial axis: The moment of inertia about radial axis, J_x, is effective over torsional frequencies. To derive its sensitivity, Eq. 24 is differentiated with respect to J_x which gives

$$\begin{bmatrix} \delta\mathbf{F}_{\mathbf{in}/p} \\ \delta\mathbf{M}_{\mathbf{in}/p} \end{bmatrix} = -\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{\mathbf{CM}/p} \end{bmatrix} \begin{bmatrix} \delta\ddot{\mathbf{x}} \\ \delta\ddot{\boldsymbol{\varphi}} \end{bmatrix} \tag{44}$$

where $\mathbf{J}_{\mathbf{CM}/p} = diag[1 \;\; 0 \;\; 0]$. If the sensitivity of the other elements of $\mathbf{J_{CM}}$ is required, similarly the value of that element can be set to unit.

NUMERICAL ANALYSES AND RESULTS

Verification

A non-rotating twisted blade [Isakson and Eisley, 1964] and a rotating blade [Wilkie et al., 1997] are analyzed to verify the linearized blade model against well-known tools. Both studies are also included in MBDyn Application manual where details can be found [Masarati, 2010]. The non-rotating twisted beam is subjected to different bending stiffness ratios in out of plane (flap) and in-plane (lag) directions ($\gamma^2 = EI_{flap}/EI_{lag}$=0, 0.01, 0.1, 0.0254). The effect of twist for linearized blade model and MBDyn is provided in Figure 3. For the rotating blade example, a beam is subjected to a range of RPM between 0 and 660. First five natural frequencies are compared with NASTRAN and MBDyn values, in the fan-plot diagram of Figure 4. Both rotating and non-rotating validation give very close results.

Sensitivity Analysis for Aeroelastic Tailoring

A sample problem is selected for a rotating blade having the properties given in Table 1. The data of the blade is loosely related to Aerospatiale Gazelle [Yamauchi et al., 1988] with a higher stiffness in chordwise direction and very stiff in radial elongation (EA) and shear deformation (GA_y and GA_z).

Table 1: Blade data used in sensitivity analysis

Main Rotor Radius	5.25 m
Blade-Hub Connection	Articulated
Flapwise Stiffness (EI_y)	0.01×10^6 Nm^2
Chordwise Stiffness (EI_z)	13×10^6 Nm^2
Torsional Stiffness (GJ)	0.03×10^6 Nm^2
Rotatory Inertia (J_p)	0.08 kgm
Mass (m)	11 kg/m
Main Rotor Angular Speed (Ω)	387 RPM

This analysis investigates the required amount of change in selected parameters as a function of a design input which produces favorable flapwise bending-torsion coupling thorough introducing elastic constant of constitutive law. As mentioned before for an isotropic and symmetric cross section, the elements of constitutive matrix $\mathbf{D}$ is diagonal. In order to introduce this coupling, elements of $\mathbf{D}$ at indices (4,5) and (5,4), $K = D_{45} = D_{54}$, are set to a non-zero value as given in Eq. 45. It should be noted that in the present work a generic coupling coefficient is considered, without analyzing how the cross section design needs to be modified in order to obtain it. Sensitivity to actual beam section design parameters will be the objective of future work.

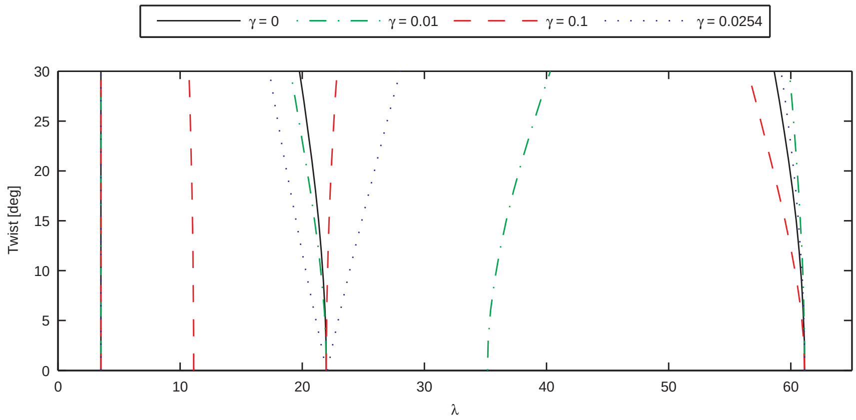
$$\mathbf{D} = \begin{bmatrix} EA & 0 & 0 & 0 & 0 & 0 \\ 0 & GA_y & 0 & 0 & 0 & 0 \\ 0 & 0 & GA_z & 0 & 0 & 0 \\ 0 & 0 & 0 & GJ & K \neq 0 & 0 \\ 0 & 0 & 0 & K \neq 0 & EI_y & 0 \\ 0 & 0 & 0 & 0 & 0 & EI_z \end{bmatrix} \tag{45}$$

The objective is to keep the natural frequencies below 5/rev constant, where 1/rev is the dimensionless rotor angular speed. This is critical since the change in blade natural frequencies leads to the change in the vibrational characteristics of the blade. The modes below 5/rev include rigid lag and rigid flap, elastic torsion and two elastic flap modes whereas the frequencies higher than 5/rev are assumed to be less important in terms of vibrational characteristics. Since tension due to centrifugal loads dominates rigid lag and flap modes and hence their dependence on elastic constants are negligible, rigid modes are not included in the formulation of sensitivity analysis. The general form of the problem can be stated as;

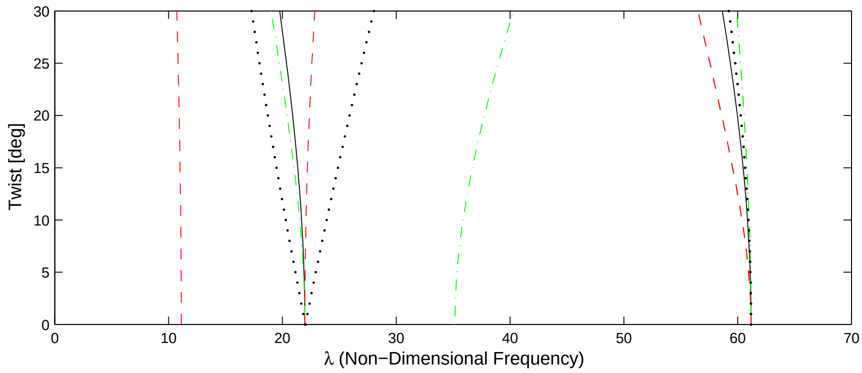
$$\sum_{i=1}^{N_p} \boldsymbol{\omega}_{/p_i} \Delta p_i = \mathbf{0} \tag{46}$$

where $\boldsymbol{\omega}_{/p_i}$ is the sensitivity of a vector of selected natural frequencies $\boldsymbol{\omega}$ with respect to change in a parameter Δp_i. The interested natural frequencies are 2nd and 3rd flap frequencies and 1st torsion.

$$\begin{bmatrix} \omega_{2F/m} \\ \omega_{3F/m} \\ \omega_{1T/m} \end{bmatrix} \Delta m + \begin{bmatrix} \omega_{2F/EI_y} \\ \omega_{3F/EI_y} \\ \omega_{1T/EI_y} \end{bmatrix} \Delta EI_y + \begin{bmatrix} \omega_{2F/J_p} \\ \omega_{3F/J_p} \\ \omega_{1T/J_p} \end{bmatrix} \Delta J_p + \begin{bmatrix} \omega_{2F/GJ} \\ \omega_{3F/GJ} \\ \omega_{1T/GJ} \end{bmatrix} \Delta GJ + \begin{bmatrix} \omega_{2F/K} \\ \omega_{3F/K} \\ \omega_{1T/K} \end{bmatrix} \Delta K = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{47}$$



(a) MBDyn



(b) Linearized Blade

Figure 3: Comparison of Non-Rotating Blade Frequencies

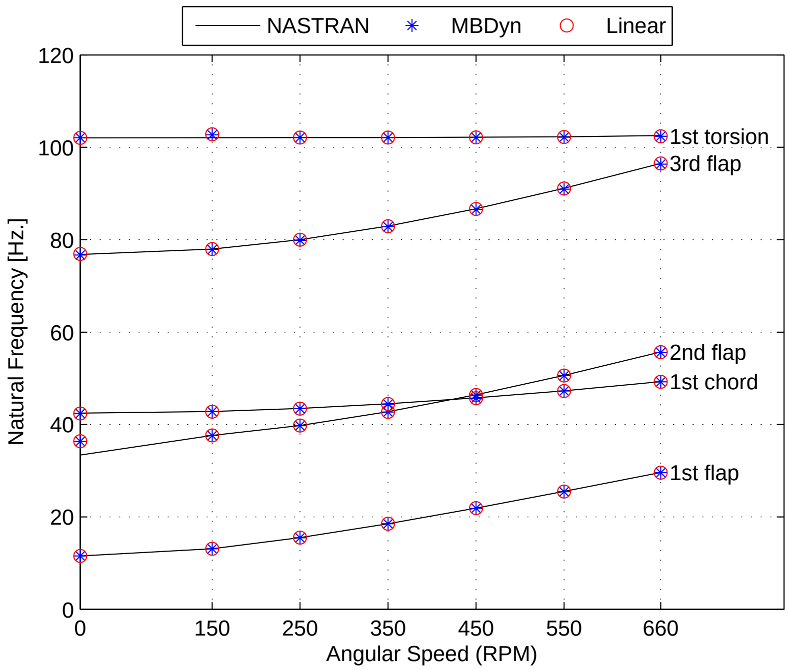


Figure 4: Fan-Plot Diagram

Since the coupling terms K is aimed to be investigated, its value and the sensitivities of natural frequencies are input, whereas the other design parameters are unknown. Then Eq. 47 can be written in matrix form as;

$$\begin{bmatrix} \omega_{2F/m} & \omega_{2F/EI_y} & \omega_{2F/J_p} & \omega_{2F/GJ} \\ \omega_{3F/m} & \omega_{3F/EI_y} & \omega_{3F/J_p} & \omega_{3F/GJ} \\ \omega_{1T/m} & \omega_{1T/EI_y} & \omega_{1T/J_p} & \omega_{1T/GJ} \end{bmatrix} \begin{bmatrix} \Delta m \\ \Delta EI_y \\ \Delta J_p \\ \Delta GJ \end{bmatrix} = - \begin{bmatrix} \omega_{2F/K} \\ \omega_{3F/K} \\ \omega_{1T/K} \end{bmatrix} \Delta K \tag{48}$$

which can also be written in closed form $\mathbf{A}\Delta\mathbf{p} = \mathbf{b}$ where $\mathbf{A}$ is the sensitivity matrix $\Delta\mathbf{p}$ is the vector of unknown parameters and $\mathbf{b}$ is the change in frequency vector due to the prescribed value of bending-torsion coupling. However as can be seen in Eq. 48, there are more unknowns ($\Delta\mathbf{p}$) than inputs ($\mathbf{b}$), hence the system in under-determined. Since the solution is not unique for such a system the required vector ($\Delta\mathbf{p}$) should be evaluated using a minimum norm solution to minimize the cost. For this purpose a norm is defined as a function of the unknown parameter vector as,

$$\Delta\mathbf{p} + \mathbf{W}^{-1}\mathbf{A}^T\Delta\boldsymbol{\lambda} = 0 \tag{49}$$

where $\mathbf{W}$ is a diagonal weighting matrix that normalizes the variables having different units with their reference values.

$$\mathbf{W} = diag[1/m_{ref}^2 \ \ 1/EI_{y,ref}^2 \ \ 1/J_{p,ref}^2 \ \ 1/GJ_{ref}^2] \tag{50}$$

In matrix form minimum norm solution is written as,

$$\begin{bmatrix} \mathbf{W} & \mathbf{A}^T \\ \mathbf{A} & 0 \end{bmatrix} \begin{bmatrix} \Delta\mathbf{p} \\ \Delta\boldsymbol{\lambda} \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix} \tag{51}$$

This is a square matrix and by partitioning Eq. 51, it can be solved for unknown $\Delta\mathbf{p}$ as

$$\Delta\mathbf{p} = \mathbf{W}^{-1}\mathbf{A}^T(\mathbf{A}\mathbf{W}^{-1}\mathbf{A}^T)^{-1}b \tag{52}$$

which can also be written as a gradient,

$$\Delta\mathbf{p}/\Delta K = -\mathbf{W}^{-1}\mathbf{A}^T(\mathbf{A}\mathbf{W}^{-1}\mathbf{A}^T)^{-1}[\omega_{2F/K} \ \ \omega_{3F/K} \ \ \omega_{1T/K}]^T \tag{53}$$

In this analysis the range of the bending torsion coupling parameter is selected as $0 \leq K \leq 0.4GJ$. In order to find the path that guarantees minimum norm and keeping the interested frequencies constant, Eq. 53 is integrated in this range with respect to K from $K = 0$ to $K = 0.4GJ$. The resulting frequencies of the design without parameter sensitivity ($\Delta\mathbf{p} = 0$) at $K = 0.4GJ$ and of the minimum norm solution ($\Delta\mathbf{p}_{min.NORM}$) at $K = 0.4GJ$ are compared with the initial design ($K = 0$) in Table 2. It can be observed that while there are significant changes between initial and $\Delta\mathbf{p} = 0$ results, the minimum norm solution preserves the initial frequencies as proposed. It can also be observed that the rigid modes, which are given as 1st lag and 1st flap in Table 2, do not change as coupling parameter K is introduced and hence the exclusion of rigid modes is a valid assumption.

Table 2: Comparison of blade natural frequencies at $K = 0.4GJ$

Frequency (/rev)	**Initial**	$\Delta\mathbf{p} = \mathbf{0.0}$	$\Delta\mathbf{p}_{min.NORM}$
1st lag	0.20	0.20	0.20
1st flap	1.02	1.02	1.02
2nd flap	2.58	2.55	2.58
3rd flap	4.44	4.25	4.44
1st torsion	4.77	4.64	4.77

The required change in parameters over this range is given in Figure 5. At any value of K, the corresponding values of four lines give the $\Delta\mathbf{p}$ vector and provides an engineering judgment about the feasibility of introduced bending-torsion coupling K. It can be observed that, the mass of the blade should be reduced while other parameters should be increased to preserve reference frequencies. Also the parameters that are related to flapping frequency, m and EI_y, are more sensitive to the problem as compared to the parameters related to torsional frequency, J_p and GJ.

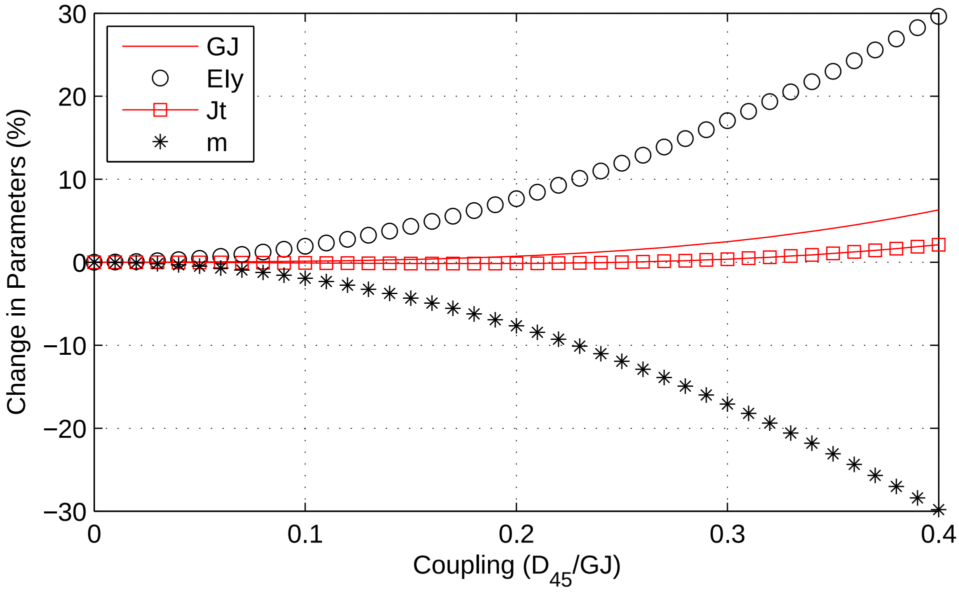


Figure 5: Sensitivity analysis for Bending-Torsion Coupling

CONCLUSIONS AND FUTURE WORK

A three node finite volume rotating blade is formulated. For the blade model, sensitivity of inertial and elasticity parameters are derived for blade natural frequencies. Sensitivity to actual beam section design parameters is planned to be included in a future work. The model is believed to provide valuable engineering judgment in the conceptual design phase of rotor blades. The aerodynamics formulation of the blade together with the sensitivity formulations are currently under development with the objective of providing efficient methods in rotorcraft aeroservoelastic design.

References

Johnson, W. (2011) *Milestones in Rotorcraft Aeromechanics*, NASA/TP2011-215971, May 2011

Johnson, W. (2011) *NDARC NASA Design and Analysis of Rotorcraft Theoretical Basis and Architecture*, AHS Aeromechanics Specialists' Meeting, Jan 2010, San Fransisco, CA

Masarati, P. (2010) *MBDyn Applications*, Dipartimento di Ingegneria Aerospaziale, Politecnico di Milano, Feb 2010

Krauskopf, B., Osinga, H., M. and Vioque, G. (2007) *Numerical Continuation Methods for Dynamical Systems*, Springer, 2007

Ghiringhelli, G. L., Masarati, P. and Mantegazza, P. (2000) *A Multibody Implementation of Finite Volume Beams*, AIAA Journal, Vol. 38(1), pp. 131-138, Jan 2000, doi:10.2514/2.933

Wilkie, W., K., Mirick, P., H., and Langston, C., W. , (1997) *Rotating Shake Test and Modal Analysis of a Model Helicopter Rotor Blade*, NASA TM4760, NASA, Jun 1997

Yamauchi, G.K. and Heffernan, R.M. and Gaubert, M. , (1988) Structural and aerodynamic loads and performance measurements of an SA 349/2 helicopter with an advanced geometry rotor, NASA TM101040

Giavotto V. and Borri M. and Mantegazza P. and Ghiringhelli G.L. and Caramaschi V. and Maffioli G.C. and Mussi F. , (1983) Anisotropic Beam Theory and Applications, Journal of Computers & Structures, Vol. 16, No. 1-4, pp. 403-413, 1983, doi:10.1016/0045-7949(83)90179-7

Isakson, G. and Eisley, J., G., (1964) *Natural Frequencies in Coupled Bending and Torsion of Twisted Rotating and Non-Rotating Blades*, NASA CR 65, NASA, July 1964