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FAVRE AVERAGED NAVIER-STOKES EQUATIONS "AGAIN"

Cahit Çıray*
Middle East Technical University, Turkey

ABSTRACT

Favre Averaging also known as Mass Averaging is another way of time averaging that can be used to obtain statistical means of quantities varying randomly in time. Here the emphasis is on a different definition of averaged quantity, though averaging process is again a regular time averaging. This averaging (for brevity Favre averaging) applied to instantaneous Navier-Stokes and related governing equations of turbulent flow of compressible fluids, show a marked advantage on corresponding fundamental equations obtained with Reynolds averaging (also regular time averaging or regular averaging in the sequel).
This presentation reminds the essentials of Favre averaging and gives the fundamental equations of the mean flow under the comprehensive title of FANS EQUATIONS (i.e. Favre Averaged Navier-Stokes) comprising equation of continuity, momentum and energy. The basic advantages relative to correponding RANS equations are shown and discussed.
Closure problem cannot be said that it is considered and no CFD solution is given.

CONTENT: Introduction- Favre averaging principle. Double and triple correlations.
FANS Equations- Continuity; Momentum; Energy
Discussion: Favre averaging against Reynolds averaging. Two energy equations. Consequences.
Interaction of modes. Can molecular re-activation be related to physics of turbulence?
Conclusion.
Note: Why "again" in the title?

INTRODUCTION

Favre averaging is an averaging on time dependency parameter of randomly varying quantities. The averaging process, so called "Favre averaging", is for turbulent flow quantities of compressible fluids as far as the author of this presentation is aware of. Its formal definition is, Favre (1965):

1 $$\overline{\rho Q} = \bar{\rho}\,\overline{\overline{Q}}$$

where "ρ" is the mass density of the compressible fluid, and "Q" is any instantaneous scalar quantity (physical properties or components of tensor quantities) of turbulent flow field of this fluid. Single overbar denotes regular time averaging and double overbar Favre averaged result. Splitting instantaneous quantities is done as follows:

2 $$Q = \overline{Q} + q' \quad \text{and} \quad Q = \overline{\overline{Q}} + q'' \qquad (\rho = \bar{\rho} + \rho' \quad \text{always})$$

where primed symbols refer to randomly fluctuating part of quantities, respectivly used in regular Reynolds (') and Favre splitting ("). It is reminded that statistical averages (time averaging in this case) are deterministic: a premise of the turbulent flow.
The basic form **1** together with splittings of **2** yields:

3 $$\overline{\rho Q} = \overline{\left(\bar{\rho} + \rho'\right)\left(\overline{\overline{Q}} + q''\right)} = \bar{\rho}\,\overline{\overline{Q}} + \overline{\rho'}\,\overline{\overline{Q}} + \overline{\rho q''} = \bar{\rho}\,\overline{\overline{Q}} + \overline{\rho q''}$$

* Ph.D., DIC. MSc, Prof. (ret.) AS Engineering Dept. Email: cciray@metu.edu.tr

The group $\overline{\rho'}\,\overline{\overline{Q}} = 0$, since with regular averaging $\overline{\rho'} = 0$. Therefore in view of the basic relation (**1**), one obtains:

4 $$\overline{\rho q''} = 0 .$$

It is remarked that in case of regular time averaging, one finds:

5 $$\overline{\rho Q} = \overline{(\overline{\rho}+\rho')(\overline{Q}+q')} = \overline{\rho}\,\overline{Q} + \overline{\rho}\,\overline{q'} + \overline{\rho'}\,\overline{Q} + \overline{\rho' q'} = \overline{\rho}\,\overline{Q} + \overline{\rho' q'}$$

Therefore, since $\overline{\rho Q}$ is unique (or from **1**):

6 $$\overline{\rho Q} = \overline{\rho}\,\overline{Q} + \overline{\rho' q'} = \overline{\rho}\,\overline{\overline{Q}}$$ and

7 $$\overline{\rho' q'} = \overline{\rho}\left(\overline{\overline{Q}} - \overline{Q}\right)$$

It is worthwhile to look at double and triple correlations multiplied by the density.
Consider two turbulence quantities Q_1 and Q_2 of a compressible fluid and the time average of their product multiplied by the fluid density. The following lines are self explanatory.

$$\overline{\rho Q_1 Q_2} = \overline{(\overline{\rho}+\rho')\left(\overline{\overline{Q_1}} + q_1''\right)\left(\overline{\overline{Q_2}} + q_2''\right)} = \overline{(\overline{\rho}+\rho')\left(\overline{\overline{Q_1}}\,\overline{\overline{Q_2}} + q_1''\overline{\overline{Q_2}} + q_2''\overline{\overline{Q_1}} + q_1''q_2''\right)} .$$ Hence:

8 $$\overline{\rho Q_1 Q_2} = \overline{\rho}\,\overline{\overline{Q_1}}\,\overline{\overline{Q_2}} + \overline{\rho q_1'' q_2''} .$$

For the same quantity RANS averaging yields:

$$\overline{\rho Q_1 Q_2} = \overline{\rho}\,\overline{Q_1}\,\overline{Q_2} + \overline{\rho}\,\overline{q_1' q_2'} + \overline{\rho' q_1'}\,\overline{Q_2} + \overline{\rho' q_2'}\,\overline{Q_1} + \overline{\rho' q_1' q_2'}$$

Similar process can be used for triple product.

9 $$\overline{\rho Q_1 Q_2 Q_3} = \overline{(\overline{\rho}+\rho')\left(\overline{\overline{Q_1}} + q_1''\right)\left(\overline{\overline{Q_2}} + q_2''\right)\left(\overline{\overline{Q_3}} + q_3''\right)} =$$

$$= \overline{(\overline{\rho}+\rho')\left(\overline{\overline{Q_1 Q_2 Q_3}} + q_1''\overline{\overline{Q_2 Q_3}} + q_2''\overline{\overline{Q_1 Q_3}} + q_3''\overline{\overline{Q_1 Q_2}} + q_1''q_2''\overline{\overline{Q_3}} + q_1''q_3''\overline{\overline{Q_2}} + q_2''q_3''\overline{\overline{Q_1}} + q_1''q_2''q_3''\right)}$$

which results in:

10 $$\overline{\rho Q_1 Q_2 Q_3} = \overline{\rho}\,\overline{\overline{Q_1 Q_2 Q_3}} + \overline{\rho q_1'' q_2''}\,\overline{\overline{Q_3}} + \overline{\rho q_1'' q_3''}\,\overline{\overline{Q_2}} + \overline{\rho q_2'' q_3''}\,\overline{\overline{Q_1}} + \overline{\rho q_1'' q_2'' q_3''} .$$

At this stage it is observed that relations like **8** and **10** contain much less number of correlations of fluctuating parts of turbulence quantities when averaging is in line with **1**.

FUNDAMENTAL EQUATIONS FOR TURBULENT FLOWS OF COMPRESSIBLE FLUIDS

Equation of continuity, momentum and energy are considered in turn.

Continuity: Instantaneous equation of continuity is:

11 $$\rho_{,t} + \left(\rho U_j\right)_{,j} = 0$$

U_j is "j" component of instantaneous velocity vector and subscripts " $_{,j}$" and " $_{,t}$" represent partial derivatives with respect to "j component of position vector **x"** and time "t", respectively. Summation convention is observed on repeating indices within a term.
Splitting with respect to **2** and averaging considering **4** yields:

12 $$\bar{\rho}_{,t} + \left(\bar{\rho}\,\overline{\overline{U}}_j\right)_{,j} = 0$$

which is the equation of continuity in FAVRE averaging of a compressible fluid. This equation does not contain any term in fluctuating components of either the density or the velocity vector. On the other hand, the conventional splitting and averaging of **11** yields:

13 $$\bar{\rho}_{,t} + \left(\bar{\rho}\,\overline{U}_j + \overline{\rho' u'_j}\right)_{,j} = 0$$

The advantage is clear.Yet, the instantaneous equation of continuity for fluctuating terms, obtained by taking the difference between **11** and **12** is**:**

14 $$\rho'_{,t} + \left(\rho u''_j\right)_{,j} + \left(\rho' \overline{\overline{U}}_j\right)_{,j} = 0$$

which is similar to the corresponding equation in terms of Reynolds splitting, i.e:

15 $$\rho'_{,t} + \left(\bar{\rho} u'_j\right)_{,j} + \left(\rho' \overline{U}_j\right)_{,j} = 0$$

Momentum equation: Familiar form of instantaneous momentum equation of a compressible fluid with negligible weight (per unit volume), written in cartesian frame is:

16 $$\rho \frac{DU_i}{Dt} = \sigma_{ij,j} = \left(-p\delta_{ij} + \tau_{ij}\right)_{,j} = -\left(p + \frac{2}{3}\mu\theta\right)_{,i} + 2\left(\mu E_{ij}\right)_{,j}$$

where

1: $$\sigma_{ij} = -\left(p + \frac{2}{3}\mu\theta\right)\delta_{ij} + 2\mu E_{ij}$$

is the stress tensor with "δ_{ij} **"** as the Kronecker delta. The viscous part of the stress tensor is:

$$\tau_{ij} = \sigma_{ij} - \left(-p\delta_{ij}\right) = -\frac{2}{3}\mu\theta\delta_{ij} + 2\mu E_{ij}\,.$$

Therefore: $$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}\,.$$

2: $-\frac{2}{3}\mu$ stands for the special second Lamé constant in view of second Stokes assumption (though a general "λ" could have been used),

3: $\theta = U_{j,j}$ stands for dilatation, i.e: time rate of relative volumetric deformation,

4: E_{ij} is the rate of strain tensor, i.e: $E_{ij} = \frac{1}{2}\left(U_{i,j} + U_{j,i}\right)$.

5: "$\rho \frac{DU_i}{Dt}$ **"** is the inertia force per unit volume and can be expressed with the help of the equation of continuity as: $$\rho \frac{DU_i}{Dt} = \left(\rho U_i\right)_{,t} + \left(\rho U_i U_j\right)_{,j}\,.$$

The left hand side of **16** can be Favre averaged and when **1** and **8** are applied the result is:

In fact, this expansion is valid for any transferable property "Q", i.e: $\rho\frac{DQ}{Dt} = (\rho Q)_{,t} + (\rho Q U_j)_{,j}$.

17 $$\overline{\rho\frac{DU_i}{Dt}} = \left(\bar{\rho}\overline{\overline{U}}_i\right)_{,t} + \left(\bar{\rho}\overline{\overline{U}}_i\overline{\overline{U}}_j + \overline{\rho u_i'' u_j''}\right)_{,j}$$

which, considering **11** becomes:

18 $$\overline{\rho\frac{DU_i}{Dt}} = \bar{\rho}\frac{D\overline{\overline{U}}_i}{Dt} + \overline{\rho u_i'' u_j''}_{,j} \quad .$$

Right hand side of **16** is reconstructed with the following premises.

1: presure is replaced by temperature with the help of the equation of state:

$p = \rho R T$ in which "R" is the gas constant and " T " is the temperature. Hence:

19 $$\bar{p} = \bar{\rho} R \overline{\overline{T}} \quad . \qquad \text{and} \qquad \overline{\rho T''} = 0$$

2: coefficient of dynamic viscosity is taken into account as:

20 $$\mu = \rho\nu \quad \text{with} \quad \bar{\mu} = \bar{\rho}\overline{\overline{\nu}} \quad \text{and} \quad \overline{\rho\nu''} = 0 \ .$$

Then, the right hand side of **16** reads in the first place as:

21 $$-\left(p + \frac{2}{3}\mu\theta\right)_{,i} + 2\left(\mu E_{ij}\right)_{,j} = -\left(R\rho T + \frac{2}{3}\rho\nu\theta\right)_{,i} + 2\left(\rho\nu E_{ij}\right)_{,j}$$

and its averaged form becomes:

22 $$-\overline{\left(p + \frac{2}{3}\mu\theta\right)_{,i} + 2\left(\mu E_{ij}\right)_{,i}} = -\left(R\bar{\rho}\overline{\overline{T}} + \frac{2}{3}\bar{\rho}\overline{\overline{\nu}}\overline{\overline{\theta}} + \frac{2}{3}\overline{\rho\nu''\theta''}\right)_{,i} + 2\left(\bar{\rho}\overline{\overline{\nu}}\overline{\overline{E}}_{ij} + \overline{\rho\nu'' E_{ij}''}\right)_{,j}$$

Therefore the Favre averaged momentum equation can be written as:

23 $$\bar{\rho}\frac{D\overline{\overline{U}}_i}{Dt} + \overline{\rho u_i'' u_j''}_{,j} = -\left(R\bar{\rho}\overline{\overline{T}} + \frac{2}{3}\bar{\rho}\overline{\overline{\nu}}\overline{\overline{\theta}} + \frac{2}{3}\overline{\rho\nu''\theta''}\right)_{,i} + 2\left(\bar{\rho}\overline{\overline{\nu}}\overline{\overline{E}}_{ij} + \overline{\rho\nu'' E_{ij}''}\right)_{,j}$$

or :

24 $$\bar{\rho}\frac{D\overline{\overline{U}}_i}{Dt} = -\left(R\bar{\rho}\overline{\overline{T}} + \frac{2}{3}\bar{\rho}\overline{\overline{\nu}}\overline{\overline{\theta}}\right)_{,i} + 2\left(\bar{\rho}\overline{\overline{\nu}}\overline{\overline{E}}_{ij}\right)_{,j} - \overline{\rho u_i'' u_j''}_{,j} - \left(\frac{2}{3}\overline{\rho\nu''\theta''}\right)_{,i} + 2\left(\overline{\rho\nu'' E_{ij}''}\right)_{,j}$$

or :

25 $$\bar{\rho}\frac{D\overline{\overline{U}}_i}{Dt} = -\left(\bar{p} + \frac{2}{3}\bar{\rho}\overline{\overline{\nu}}\overline{\overline{\theta}}\right)_{,i} + 2\left(\bar{\mu}\overline{\overline{E}}_{ij}\right)_{,j} - \overline{\rho u_i'' u_j''}_{,j} - \left(\frac{2}{3}\overline{\rho\nu''\theta''}\right)_{,i} + 2\left(\overline{\rho\nu'' E_{ij}''}\right)_{,j}$$

Energy equation: The instantaneous differential equation of energy, without any direct heat added or taken out during the process, may be expresed in two different forms:

A: Direct application of the first law of thermodynamics leads to:

26 $$\rho\frac{D}{Dt}\left(e + \frac{U_j^2}{2}\right) = \left(\sigma_{ij}U_i\right)_{,j} + \left(kT_{,j}\right)_{,j}$$

B: The latter equation can be organized to have on the left hand side the substantial derivative of total enthalpy. One ends with the familar form of the energy equation**:**

27 $$\rho\frac{Dh_T}{dt} = p,_t + \left(\tau_{ij}U_i\right),_j + \left(kT,_j\right),_j$$

where:

total enthalpy $$h_T = e + \frac{p}{\rho} + \frac{U_j^2}{2} = h + \frac{U_j^2}{2} = C_P T + \frac{U_j^2}{2}$$

with "h" as the static enthalpy, "e" internal energy (both per unit mass) and " C_P " as the specific heat of the gas at constant pressure.

Each of these two energy equations (i.e: **26** and **27**) is treated below.

A: 26 can be reduced to a more managable form:

since: $$\rho\frac{D}{Dt}\left(e+\frac{U_i^2}{2}\right) = \rho\frac{De}{Dt} + \rho\frac{D}{Dt}\left(\frac{U_i^2}{2}\right)$$; then: $$\rho\frac{De}{Dt} + \rho U_i\frac{DU_i}{Dt} = \left(\sigma_{ij}U_i\right),_j + \left(kT,_j\right),_j$$

With the help of **16,** and decomposition of the terms following the third equality sign, one obtains:

$$\rho\frac{De}{Dt} + \sigma_{ij},_j U_i = \sigma_{ij},_j U_i + \sigma_{ij}U_i,_j + \left(kT,_i\right),_i$$

The ultimate result is:

28 $$\rho\frac{De}{Dt} = \sigma_{ij}U_i,_j + \left(kT,_i\right),_i = -p\theta + \tau_{ij}U_i,_j + \left(kT,_i\right),_i$$

Right hand side of the first equality sign is expanded as:

$$\sigma_{ij}U_i,_j + \left(kT,_j\right),_j = \left[-\left(p+\frac{2}{3}\mu\theta\right)\delta_{ij} + 2\mu E_{ij}\right]U_i,_j + \left(kT,_i\right),_i = -\left(p+\frac{2}{3}\mu\theta\right)\theta + 2\mu U_i,_j E_{ij} + \left(kT,_i\right),_i$$

But the part of the viscous dissipation term containing rate of strain can further be simplified as:

$$U_i,_j E_{ij} = \left(E_{ij}+\omega_{ij}\right)E_{ij} = E_{ij}^2$$

since $\omega_{ij}E_{ij} = 0$, where " ω_{ij} " is the unsymmetrical part of the velocity gradient tensor " $U_i,_j$ ". Hence

the energy equation in the form of **26** becomes:

29 $$\rho\frac{De}{Dt} = -\left(p+\frac{2}{3}\mu\theta\right)\theta + 2\mu E_{ij}^2 + \left(kT,_i\right),_i = -p\theta + \left(kT,_i\right),_i + \mathbf{\Phi}$$

Here $$\mathbf{\Phi} = -\frac{2}{3}\mu\theta^2 + 2\mu E_{ij}^2 = -\mu\left(\frac{2}{3}\theta^2 - 2E_{ij}^2\right)$$

is the viscous dissipation term.

The splitting of **26** according to Favre and averaging yields:

30 $$\overline{\rho\frac{De}{Dt}} = \left(\overline{\rho e}\right),_t + \left(\overline{\rho e U_j}\right),_j = \left(\bar{\rho}\,\overline{\overline{e}}\right),_t + \left(\bar{\rho}\,\overline{\overline{e}}\,\overline{\overline{U}}_j + \overline{\rho e'' u_j''}\right),_j = \bar{\rho}\frac{\overline{\overline{D}}e}{Dt} + \left(\overline{\rho e'' u_j''}\right),_j$$

for the left hand side. The option of replacing "e" in terms of "T", i.e: $e = C_V\, T$ and $e'' = C_V\, T''$, can be done directly; since such a change will not alter the equation and will not provide additional correlation term(s) if C_V is taken as a constant.

The instantaneous right hand side is : $-\left(p+\frac{2}{3}\mu\theta\right)\theta + 2\mu E_{ij}^2 + \left(kT,_i\right),_i$

In the following treatment, **19** and **20** are observed; hence the average of the right hand side of **29** becomes:

31
$$\overline{-\left(p+\frac{2}{3}\rho\nu\theta\right)\theta+2\rho\nu E_{ij}^2+\left(kT,_j\right),_j} = -\left(R\bar{\bar{T}}+\frac{2}{3}\bar{\bar{\nu}}\bar{\bar{\theta}}\right)\bar{\rho}\bar{\bar{\theta}} + 2\bar{\rho}\bar{\bar{\nu}}\bar{\bar{E}}_{ij}^{2} + \left(\bar{k}\bar{\bar{T}},_j\right),_j -$$
$$-R\overline{\rho T''\theta''} - \frac{4}{3}\bar{\bar{\theta}}\,\overline{\rho\nu''\theta''} - \frac{2}{3}\bar{\bar{\nu}}\,\overline{\rho\theta''^2} - \frac{2}{3}\overline{\rho\nu''\theta''^2} + 2\left(\bar{\bar{\nu}}\,\overline{\rho E_{ij}''^2} + 2\bar{\bar{E}}_{ij}\overline{\rho\nu'' E_{ij}''} + \overline{\rho\nu'' E_{ij'}''^2}\right)$$

Therefore Favre averaged form of the Energy equation is:

32
$$\bar{\rho}\frac{D\bar{\bar{e}}}{Dt} = -\left(\bar{p}+\frac{2}{3}\bar{\mu}\bar{\bar{\theta}}\right)\bar{\bar{\theta}} + 2\bar{\mu}\bar{\bar{E}}_{ij}^{2} + \left(\bar{k}\bar{\bar{T}},_j\right),_j - R\overline{\rho T''\theta''} - \frac{4}{3}\bar{\bar{\theta}}\,\overline{\rho\nu''\theta''} - \frac{2}{3}\bar{\bar{\nu}}\,\overline{\rho\theta''^2} +$$
$$+2\left(\bar{\bar{\nu}}\,\overline{\rho E_{ij}''^2} + 2\bar{\bar{E}}_{ij}\overline{\rho\nu'' E_{ij}''}\right) - \frac{2}{3}\overline{\rho\nu''\theta''^2} + 2\left(\overline{\rho\nu'' E_{ij'}''^2}\right) - \left(\overline{\rho e'' u_j''}\right),_j$$

B. The alternative energy equation **27** may also be treated in a similar way.

27 (repeat**)**
$$\rho\frac{Dh_T}{dt} = p,_t + \left(\tau_{ij}U_i\right),_j + \left(kT,_j\right),_j$$

Left hand side is:

$$\rho\frac{Dh_T}{dt} = \rho\frac{D}{Dt}\left(h+\frac{U_i^2}{2}\right) = \rho\frac{Dh}{Dt} + \rho U_i\frac{DU_i}{Dt} = \rho\frac{Dh}{Dt} + U_i\sigma_{ij},_j$$

and

$$\rho\frac{Dh}{Dt} + U_i\sigma_{ij},_j = \rho\frac{Dh}{Dt} - U_i p,_i + U_i\tau_{ij},_j$$

expanding the right hand side :

$$\rho\frac{Dh}{Dt} - U_i p,_i + U_i\tau_{ij},_j = p,_t + \left(\tau_{ij}U_i\right),_j + \left(kT,_j\right),_j = p,_t + U_i\tau_{ij},_j + \tau_{ij}U_i,_j + \left(kT,_j\right),_j$$

yields **27** as:

33
$$\rho\frac{Dh}{Dt} = \frac{Dp}{Dt} + \tau_{ij}U_i,_j + \left(kT,_j\right),_j$$

The last equation will not be considered for further development as explained below (item 2 of **Disccussion**).

SOME DISCUSSION

1: The set of equations composed of continuity, momentum and energy are referred as fundamental equations to distinguish them from governing equations of turbulent flows comprising closure formulations and other auxilliary relations in addition to fundamental equations.

2: Equation **33** does not express any advantage compared to **29.** Contrary, both sides of **33** contain total derivative of pressure in some form, which is a repetition from physical point of view, therefore an unnecessary burden for calculations. Hence, **33** is not carried further.

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On the other hand, arguments related to **29** forwarded in the sequel, could have been repeated for **33,** **(**if for some reason there is interest).

3: FAVRE averaging leading to FANS equations composed of **12, 25, 32** are fundamental equations for the study of linear Stokes fluids. During the derivation of these equations, Stokes second assumption leading to the definition of second Lamé constant "λ", i.e. the compressibility constant in terms of dynamic viscosity, is used . This is done just to help an easy comparison with RANS or NS equations developed commonly with the same premise for incompressible fluids.
Similar argument is used when perfect gas relation is introduced to relate pressure to temperature via mass density. If the study warrants, these simplifications can be replaced by more accurate and suitable relations. In the same spirit, calorically perfect gas assumption will be used when discussing further simplifications of fundamental equations.
It is to be noted that these considerations do not hamper the advantage of FAVRE averaging procedures.

4: The reduction in the number of correlations is surely the advantage of FAVRE averaging.This can be seen by comparing the number of scalars defining tensors quantities in RANS and FANS equations due to fluctuations.

TABLE

Number of independent scalars* appearing in correlations for 3D flows

EQUATION(s)	RANS	FANS
Continuity (**12**)	1X3 =3	–
Momentum (**25**)	1X15+1X10+27= 52	1X6+1X3+1X6 = 15
Energy (**32**)	1X4X3+1X4X3+1X15+ +1X15+1x6+1X15+1X15= 90	1X3+1X3+1X13+1X12+ +1X6+1X3+1X12+1X3 = 45

Another avenue of comparison is possible.

A: The Favre averaged form of the equation of continuity admits a correct (i.e: physically consistent) definition of streamline without any flow across the streamline. On the other hand , Reynolds averaging of the same equation contains the additional term of:

$$\overline{\rho' u_i'},_i$$

which does not confirm the definition of streamline based on mean velocity resulting from Reynolds averaging, Lele (1994).

B: The associated term $\overline{\rho' u_i'}$ appers as a source term: this point wise source term becomes positive or negative depending on the flow conditions. A source term of mass may indicate that the mathematics does not represent the physics adequately, Spina et al.(1994).

Further discussion of relative merits or disadvantages of FAVRE averaging vs. Reynolds averaging can be found in Lele (1993).

C: The number of correlations that for one or other reason is omitted from the equations is less in Favre averaging. Then, it is possible to say that mass averaging reflects the physics of turbulent behaviour adequately.
.
5: The fundamental equations derived so far, i.e: **12, 25** and **32**, are exact to the extent that the instantaneous equation are exact. In spite of the fact that FANS fundamental equations have a marked advantage in terms of complexity when compared to equivalent RANS equations, a simplification is still warranted perhaps to cause little (negligible) deviation from the more exact nature of FANS equations.
This simplification is based on the assumption:

* Numerical terms in this table are in the form of aXb where the symbols mean:

a: how many times the term appears in the equation
b: number of independent scalars defining the term, counted in fluctuating primitive variables: ρ, T'', U_i'' .

$$\overline{\rho\nu''q''} = \overline{\rho\nu''q_1''q_2''} = 0$$

This means that one point double and triple correlations containing the product (ρν’’) can be neglected. The fact that $\overline{\rho\nu''}$ = 0, may be thought to contribute to this end. Then, the number of correlations in momentum (**25**) and energy (**32**) equations are further reduced. The whole set is reproduced below for the sake of completeness:

12 (repeat) $$\bar{\rho},_t + \left(\bar{\rho}\overline{\overline{U}}_j\right),_j = 0$$

34 $$\bar{\rho}\frac{D\overline{\overline{U}}_i}{Dt} = -\left(\bar{p} + \frac{2}{3}\bar{\rho}\overline{\overline{\nu\theta}}\right),_i + 2\left(\bar{\mu}\overline{\overline{E}}_{ij}\right),_j - \overline{\rho u_i'' u_j''},_j$$

35 $$\bar{\rho}\frac{D\overline{\overline{e}}}{Dt} = -\left(\bar{p} + \frac{2}{3}\bar{\mu}\overline{\overline{\theta}}\right)\overline{\overline{\theta}} + 2\bar{\mu}\overline{\overline{E}}_{ij}^{\,2} + \left(\bar{k}\overline{\overline{T}},_j\right),_j - R\overline{\rho T''\theta''} - \frac{2}{3}\overline{\overline{\nu\rho\theta''^2}} + 2\left(\overline{\overline{\nu\rho E_{ij}''^2}}\right) - \left(\overline{\rho e'' u_j''}\right),_j$$

6+21= 27 independent scalars form the correlations appearing in equations **34** and **35** instead of 18+45= 63 which appear in **25** and **32**. No doubt, similar argument can be used for any correlation in **35** provided that the simplification is justified with respect to the particular problem in hand.

It becomes clear that the fundamental equations for compressible fluids in Favre averaged form approach turbulent flow equations of incompressible fluid (though with obvious differences), to the extent that simplification mentionned above is admitted.

The time independancy of viscosity is generally accepted for momentum and energy equations See KUO (1986),(though energy equation is in the form of **33).**

6: Time correlations between viscosity, density (shown above), temperature, velocity, pressure are discussed with the intention whether to keep them in or remove from the set of fundamental equations. Therefore correlations representing interaction between these quantitites have been the subject of intense studies. The phenomenon of interaction was introduced by Professor M. A. Morkovin in the colloque of 1961 organized by Professor A. Favre, at Marseille.
Professor Kowasznay grouped various interactions with respect to three main quantity or “mode”. The preference is exhibited as to use instead of velocity, temperature and density, the “more analytically expressible terms” of vorticity, entropy (or total temperature) and pressure, Bradshaw P. (1977); (the paper of Chu and Kowazsnay referred by Bradshaw, dates from 1958). So these interactions are discussed around vorticity, entropy and pressure and the phenomenon is commonly named the “intereaction of modes”
The subject is reported in various papers and reviews, see: Bradshaw P. (1977), Cebeci T. et al. (1984), Lele S. K.(1994), Spina, E.F. et al. (1994).

*A brief, hence incomplete summary, related mostly to boundary layers, is attempted here within item **6**.*

A: The very first and important observation is that: turbulence fluctuations Mach number **M’** (i.e. the RMS of the dıfference between the instantaneous Mach number and its mean value) satisfies:

M’ < 0.15 ~ 0.20 if $\mathbf{M_e}$ < 5.00

as stated by Morkovin (1961). Here $\mathbf{M_e}$ stands for the Mach number at the edge of the boundary layer. Yet, Spina (1994) produces the **Figure 1**, on the basis of more recent measurements, which shows that:

M’ < 0.3 if $\mathbf{M_e}$ < 5.00 range

The consequence is that ”*the turbulence structure within the boundary layer is not appreciably influenced by compressibility of the fluid”*.

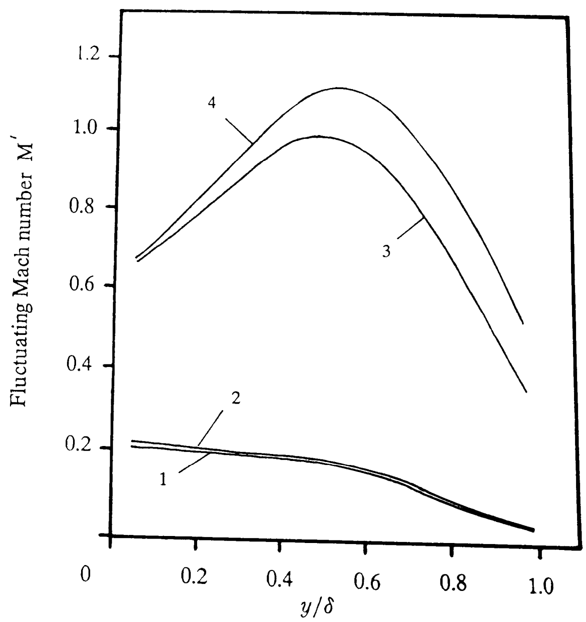

Figure 1: *Fluctuating Mach Number Distribution.* ***Flow 1:*** $M_e = 2.32$, *R_e = 4700, adiabatic wall (Elena & Lacharme,1988);* ***Flow 2:*** $M_e = 2.87$, *R_e = 80000, adiabatic wall (Spina & Smits, 1987);* ***Flow 3:*** $M_e = 7.2$, *R_e = 7100, T_w/T_e = 0.20 (Owen & Horstman, 1972);* ***Flow 4:*** $M_e = 9.4$, *R_e = 40000, T_w/T_e = 0.40 (Laderman & Demetriades, 1974) . Figure from : Spina et al. (1994).*

It is remarked that the limit of similarity of turbulence structure of compressible mixing layers and jets to the in-compressible counterparts is ≈ 1.5, Bradshaw (1977), far smaller than the corresponding value 5 valid for Boundary Layers.
The discussions on turbulent boundary layers of compressible fluids is based on a pair of observation by Morkovin, Morkovin (1961). Ensuing developments are repetitions of what can be found in Morkovin (1961), Bradshaw (1977), Cebeci (1984):

A: The first observation is that the ratio of fluctuating local pressure to local mean pressure is small. Then,using equation of state, one may write:

$$\frac{p'}{\overline{p}} = \frac{\rho'}{\overline{\rho}} + \frac{T'}{\overline{T}} \approx 0 . \quad \text{Hence} \quad \frac{\rho'}{\overline{\rho}} = -\frac{T'}{\overline{T}}$$

B: The second observation is that, the fluctuations of total temperature (T_T') are small, i.e. smaller than the temperature fluctuations, **Figure 2.** Then starting with the definition of total temperature one arrives to its fluctuations, which reads:

$$T_T' = T' + \frac{1}{C_p}\overline{U}u' \approx 0 . \quad \text{Hence} \quad \frac{\rho'}{\overline{\rho}} = -\frac{T'}{\overline{T}} = \frac{1}{C_p\overline{T}}\overline{U}u'$$

The last relation can be put in a formal form as:

36
$$\frac{\rho'}{\overline{\rho}} = -\frac{T'}{\overline{T}} = (\gamma-1)\,\mathbf{M}^2\frac{u'}{\overline{U}}$$

One must note that **M** is the local Mach number and $\mathbf{M_e} = \mathbf{M}\frac{\overline{U_e}}{\overline{U}}$.

The relation **36** is physically plausible for *high speed* boundary layers if the local Mach number **M** < 1 in which the velocities, hence its fluctuations are small and in opposite variability with temperature fluctuations, **36.**

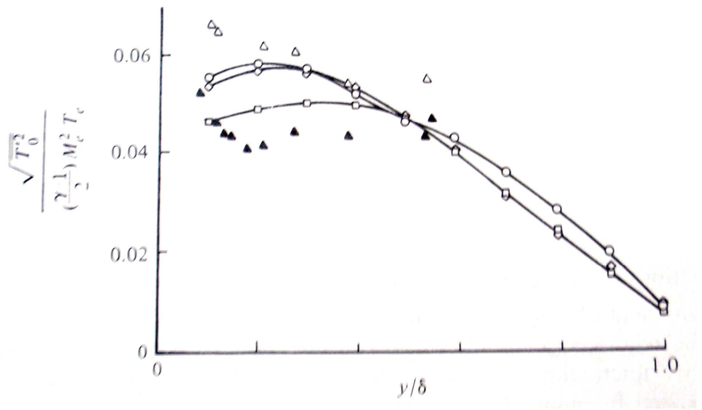

Figure 2: *Non Dimensionalized RMS Fluctuations of Total Temperature, Cebeci (1984).*

*The ordinate in the original of this **Figure** from Morkovin (1961, p:379) is:* $\sqrt{\overline{T_T'^2}}$ **:** $\left(\overline{T_{Te}} - \overline{T_e}\right)$, *and the numerical scale does not change. Morkovin also indicates that the symbols in the **Figure** are data from supersonic Boundary Layers with* M_e ***= 1.77*** *(Morkovin-Phinney) and* M_e***= 1.72; 3.56; 4.67****(all from Kistler).* T_0' *of the Figure is* T_T'.

On the other hand,with heated walls, temperature distribution is controlled by wall temperature (more correctly by: T_w–T_e) and not by velocity distribution as was the case for high speed flows. Hence **36** is not valid for low speed flows on heated walls. Meanwhile, the consequence of the first observation is still valid. As a second step the so called SRA (Strong Reynolds Analogy), first proposed and formulated by Professor Alec D. Young, Young (1951) is used. (The name SRA was later given by Professor Morkovin (1961), p:374). This step corresponds to write:

$$\frac{T'}{T_w - T_e} = \frac{u'}{U_e}$$

which when combined with the consequence of the first observation, yields:

37 $$\frac{\rho'}{\overline{\rho}} = -\frac{T'}{\overline{T}} = \left(\frac{T_w - T_e}{\overline{T}}\right)\frac{u'}{U_e}$$

So , this relation is good for low speed flows, if Pr ≈ 1.

7: One may ask how these elaborations (**36, 37)** developed with respect to Reynolds averaging for turbulent boundary layers are reflected on Favre averaged quantitites. We have to start with another principal relation in the same line as **7.** The following development is mathematically correct:

$$q'' = Q - \overline{\overline{Q}} \;;\; \overline{\rho}q'' = \overline{\rho}Q - \overline{\rho}\overline{\overline{Q}} \;;\; \overline{\overline{\rho}q''} = \overline{\overline{\rho}Q} - \overline{\rho}\overline{\overline{Q}} \;;\; \overline{\overline{\rho}q''} = \overline{\rho}\left(\overline{Q} - \overline{\overline{Q}}\right).$$

Hence in view of **7** one obtains:

38 $$\overline{\rho'q'} = \overline{\rho}\left(\overline{\overline{Q}} - \overline{Q}\right) = -\overline{\overline{\rho}q''} \longrightarrow \overline{q''} = -\frac{\overline{\rho'q'}}{\overline{\rho}}$$

A: This result shows that in general the magnitude in terms of Favre fluctuations ('') are in the ratio of $\approx \frac{\rho'}{\overline{\rho}}$ as compared to Reynolds fluctuations. For double correlations take the following example:

$$\overline{\rho u_i'' T''} \approx \overline{\rho \frac{\rho'}{\bar{\rho}} u' \frac{\rho'}{\bar{\rho}} T'} = \overline{\left(\frac{\rho'}{\bar{\rho}}\right)^2 \rho u_i' T'} \approx \frac{\overline{\rho'^2}}{\bar{\rho}^2} \overline{\rho u_i' T'}$$

For supersonic boundary layers, one can see that Favre fluctuations are one order of magnitude smaller than the corresponding Reynolds fluctuations and the double correlations are two order of magnitude smaller. This shows that, in case some correlations are neglected, this neglect is more justifiable in case of FANS Equations.

B: For non- hypersonic high speed boundary layers, one can start from the relation:

$$T' = -\frac{1}{C_p}\bar{U}u' \text{ . Then : } \rho' T' = -\frac{1}{C_p}\bar{U}u'\rho' \text{ and : } \overline{\rho' T'} = -\frac{1}{C_p}\bar{U}\overline{\rho' u'}$$

The last relation, with **38** becomes: $\overline{T''} = -\frac{1}{C_p}\overline{\bar{U}u''}$. One may even infer that:

$$T'' \approx -\frac{1}{C_p}\bar{\bar{U}}u''$$, a relation almost the same as the starting one.

Such a conclusion means that double correlations and the term $\left(\overline{\rho e'' u_j''}\right)$ appearing in energy equation of the form **35** can be neglected.

C: An aproximate relation between Favre defined correlation of order "n" and Reynolds defined corresponding one can be found using **38.** The result is:

39 $$\overline{\rho q_1'' q_2'' \ldots\ldots q_n''} \approx (-1)^n \bar{\rho}\, \overline{q_1' q_2' \ldots\ldots q_n'} \left[\overline{\left(\frac{\rho'}{\bar{\rho}}\right)^n} + \overline{\left(\frac{\rho'}{\bar{\rho}}\right)^{n+1}}\right]$$

The second term in the bracket is one order smaller than the first one. It can be neglected if need be, depending on the magnitude of $\frac{\rho'}{\bar{\rho}}$.

This relation also shows that the magnitude of Favre defined correlations decrease quickly as the order of the correlations increase.

8: In the hypersonic range, the relation **36** $\frac{\rho'}{\bar{\rho}} \sim \mathbf{M^2}$**,** suggests that double correlations cannot be neglected and the correlation must be individually considered with respect to flow conditions and it is not attempted here.

9: It is of interest to ask whether fluctuations of viscosity and of other material properties of fluid do follow temperature fluctuations faithfully, and if temperature fluctuations do also follow velocity fluctuations instantaneously, or are there some phase lags between these quantities introducing some sort of amplification or damping effect to fluctuations with natural consequences on correlations?

Perhaps, it may be worthwhile to consider the interaction of modes from a different angle.

A:Take point wise quantities such as temperature, pressure, density, viscosity etc. They are all quantities defined at macroscale but are generated in different size of volumes from the same or similar activities of molecules and atoms, hence at microscale. The interaction between them must be through a propagation process, hence with a delay and phase shift. The delay between oscillating quantities may lead to damping.
If damping occurs, two consequences are in place:

- Correlations on same quantities (modes) are not expected to have a damping of this sort, (ie: $-\overline{\rho u_i'' u_j''}$)
- Correlations of different quantites may be subjected to a damping of this sort, (ie: $\overline{\rho T'' \theta''}$). If this is true, it may explain the negligible magnitude of the correlations of different modes for non-hypersonic boundary layers.

It appears that provided the boundary layer flow is not hypersonic (i.e: **M** *< 5) self correlation of vortical mode is enough to be considered in the governing Favre equations of compressible turbulent flow.*

B: Lele (1993) begins his review with the sentence emphasizing that “Turbulence is a macroscopic state of flow in which …..”. This statement which forms one of the pedestal of turbulence studies is a consequence of our **perception** of turbulence and cannot be denied. Yet, one has to ask whether such a perception does not limit the necesssary (or missing) elements to understand the physics of turbulence and restricts our minds to go to other time and spatial scales where the root(s) of turbulence may prevail. Let us continue with another scenario.

The passage of energy from macroscale re-activates the molecules. Thus molecules are energized. Depending on their limitations of activity, their saturation level of absorbtion (of energy), the remainder of the energy bounces back and propagates to macroscale observable activities.

At microscale some activities are in vibrating state. The re-activation reinfoces these vibrations. During the propagation, the amplitude of these vibrations can increse or damp down depending on the capability of the macrocosmos to assimilate the incoming energy.
Ultimately this sort of bounced back energy may be damped or not at macroscale. If they are not damped, they will likeley cause and maintain turbulence.

The study of such a structure will involve more physical (material) elements like molecules, atoms, mean free paths and associated velocities, compaction, collision energies, heat formation volumes, density formation volumes, viscosity formation volumes, propagation velocities, instability propagation emanating from disturbed molecular activity to enhance macroscale disorder in the form of turbulence, etc….. Kinetic theory of gases may be of help.

C: Finally. one may write the instantaneous velocity of the molecule to any complexity consistent with our knowledge of the molecular activity, but comprising the macroscale velocity and also microsacale ones. The average of the square of this velocity will contain macroscale energy items as well microscale ones (such as those that give rise to heat we measure at macroscale). Some of the terms forming this expression are:

correlations between micro scale and macro scale activities.
One may infer that each of these activities (micro and macro scales) influence the other, and in both direction.

CONCLUSION

1: The use of FANS equations are more accurate and economical.This conclusion of the author is not always shared.

2: The use of energy equation in terms of internal energy is preferable to its total enthalpy version.

3: This article says that to stick to RANS or FANS equations will not be sufficient to understand the physics of turbulence fully; more, to find common roots and formulations to solve turbulent flows. Microcosmos may give some clues in this direction.

ACKNOWLEDGEMENT AND WHY “AGAIN”

This paper is in part a summary of author’s learning process. Unfortunately he did not see much about FANS equations when he was concerned with turbulence, perhaps mostly with incompressible one. So it was his desire to attract the attention to FANS equations. Hence his learning process began. At the same time he submitted an abstract to AIAC 2013 on the subject. In due course, he observed that some of his conclusions related to FANS equations were already realized by other people. Therefore he withdrew the abstract from AIAC 2013.
There, the author met the objection of Professor I. H. Tuncer. The author must humbly admit his incompetance to surmount Professor I. H.Tuncer’s will. So he decided to curb at least the boldness of the original title with the present one.

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Note on references. The review papers quoted from ARFM contain dozens of references related to the subject.